Critical Phenomena in Gravitational Collapse

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Critical Phenomena in gravitational collapse

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Universality and Scaling in Gravitational Collapse of a Massless Scalar Field

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(Received 22 September 1992)

I summarize results from a numerical study of spherically symmetric collapse of a massless scalar field. I consider families of solutions, $\mathcal{S}[p]$, with the property that a critical parameter value, $p^\star$, separates solutions containing black holes from those which do not. I present evidence in support of conjectures that (1) the strong-field evolution in the $p \to p^\star$ limit is universal and generates structure on arbitrarily small spatiotemporal scales and (2) the masses of black holes which form satisfy a power law $M_{\rm BH} \propto |p - p^\star|^\gamma$, where $\gamma \approx 0.37$ is a universal exponent.

A numerical experiment...

- Consider scalar wave

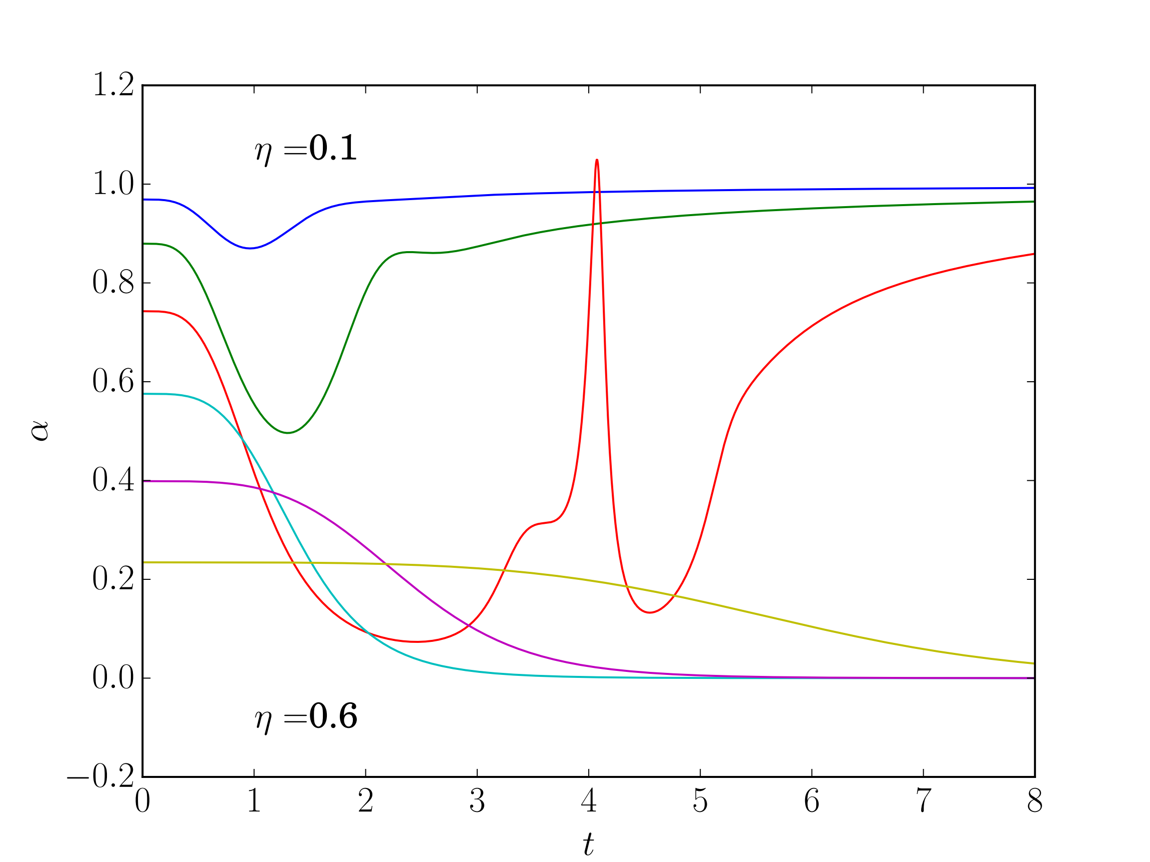
$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



Have *critical value* η_* so that

$$\eta < \eta_* \qquad \alpha \to 1 \qquad \text{end up with flat space}$$

$$\eta > \eta_* \qquad \alpha \to 0 \qquad \text{end up with black hole}$$

Black-hole threshold

$$0.3 < \eta_* < 0.4$$

A numerical experiment...

- Let's say scalar wave

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta \exp(-R^2/R_0^2)$$

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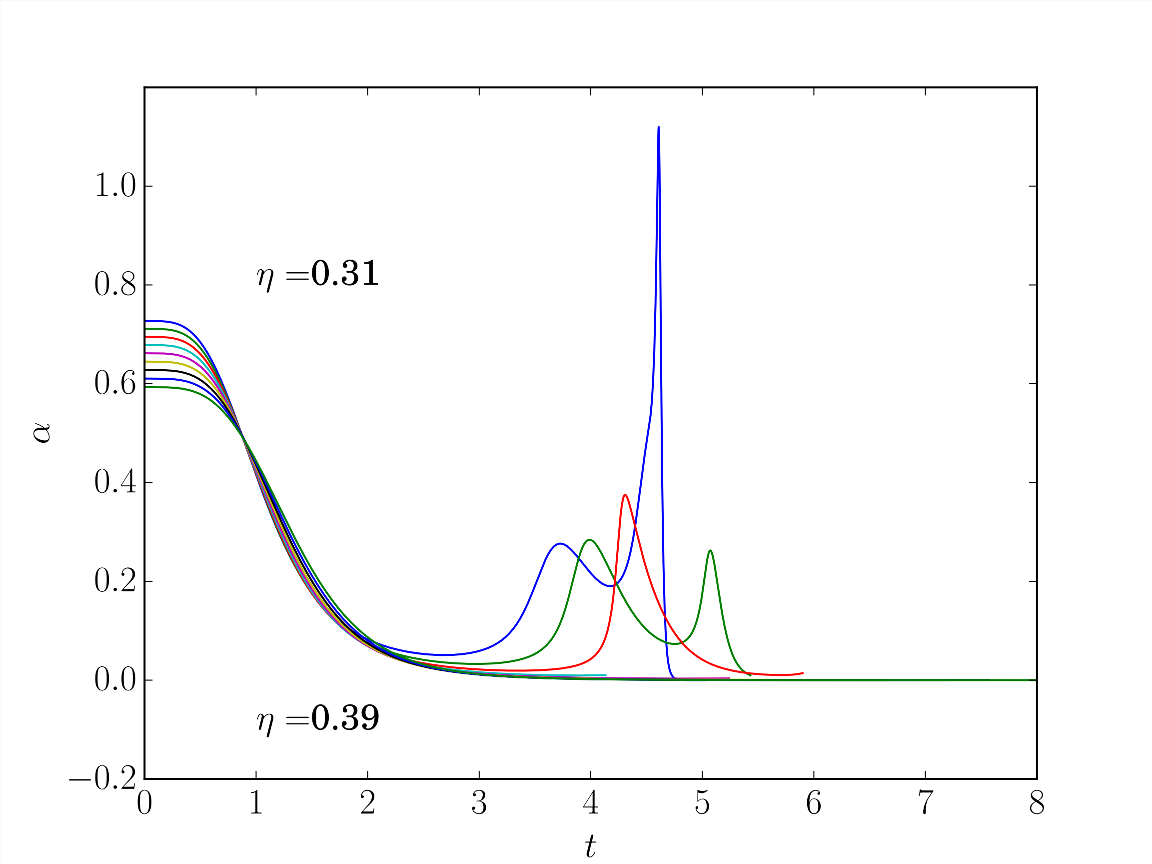

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Black-hole threshold

$$0.3 < \eta_* < 0.31$$

A numerical experiment...

- Let's say scalar wave

$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

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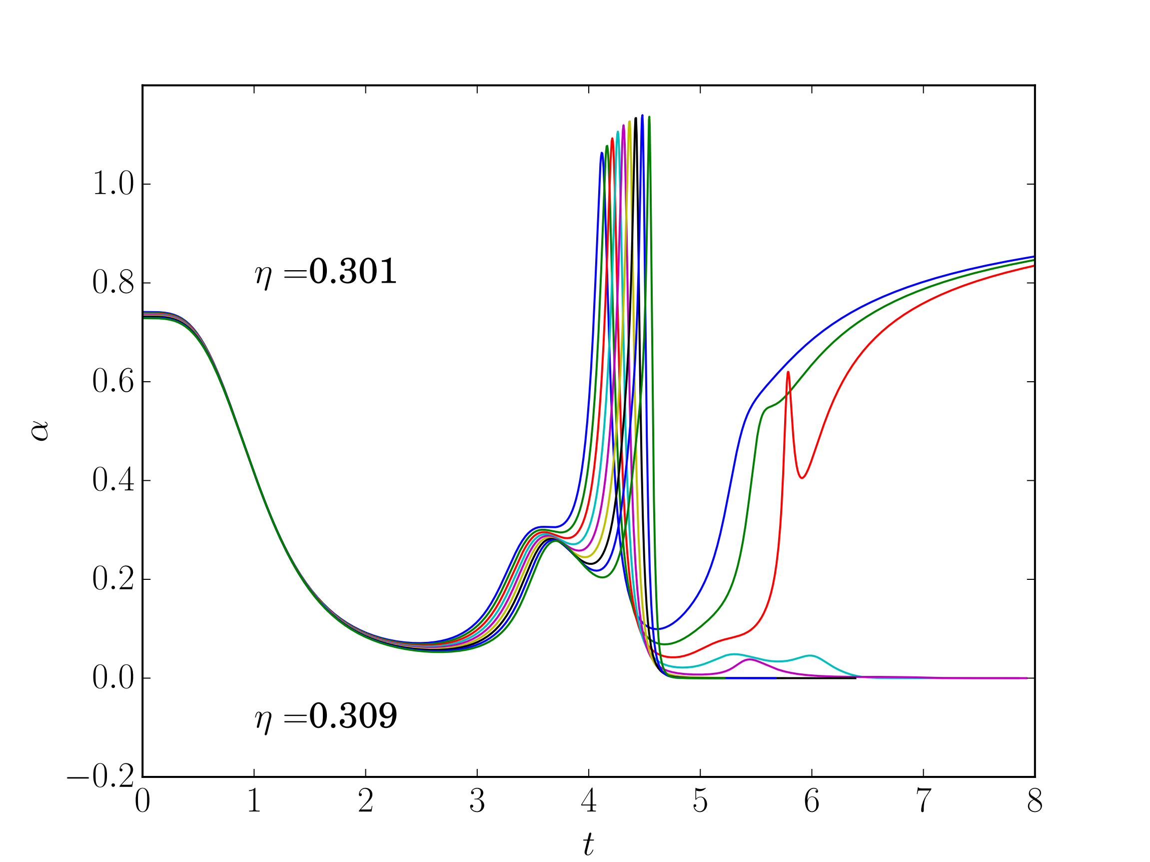

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$$\eta < \eta_* \qquad \alpha \to 1 \qquad \text{end up with flat space}$$

$$\eta > \eta_* \qquad \alpha \to 0 \qquad \text{end up with black hole}$$

Black-hole threshold

$$0.303 < \eta_* < 0.304$$

A numerical experiment...

- Let's say scalar wave

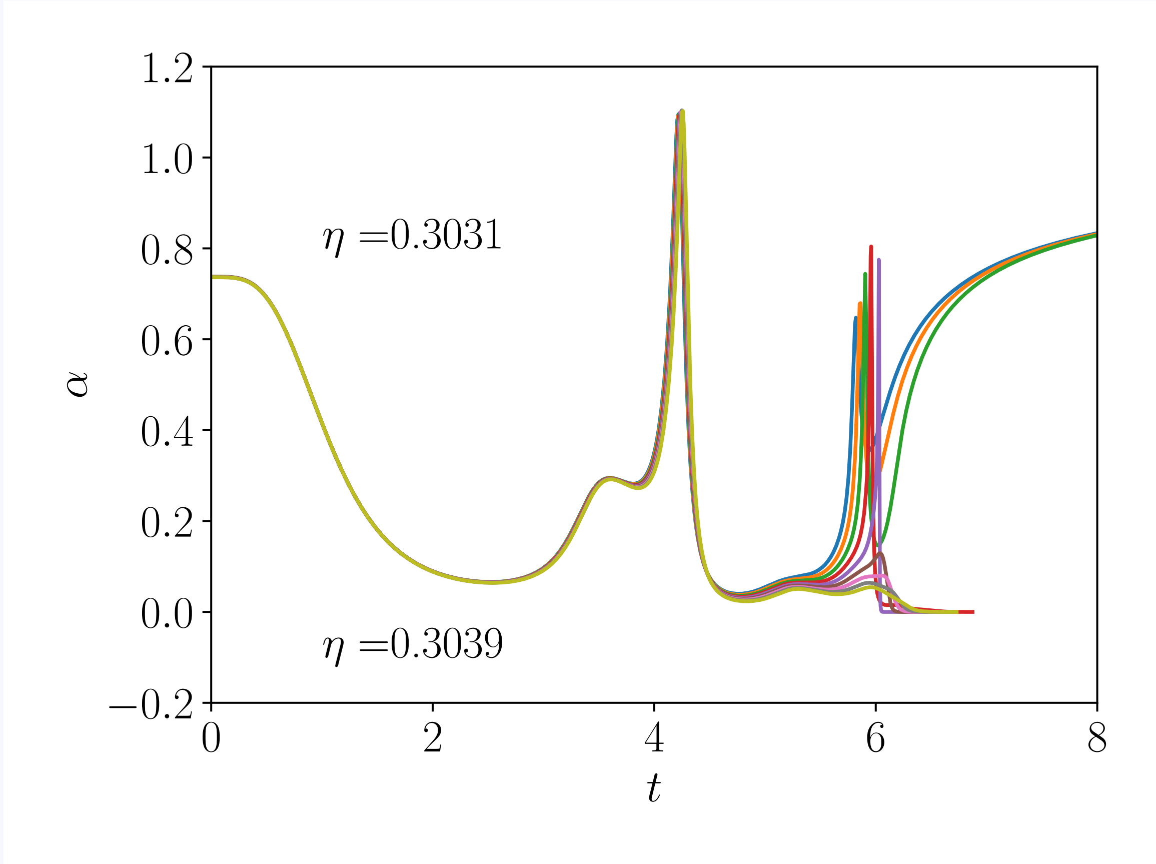
$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



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Black-hole threshold

$$0.3033 < \eta_* < 0.3034$$

A numerical experiment...

- Let's say scalar wave

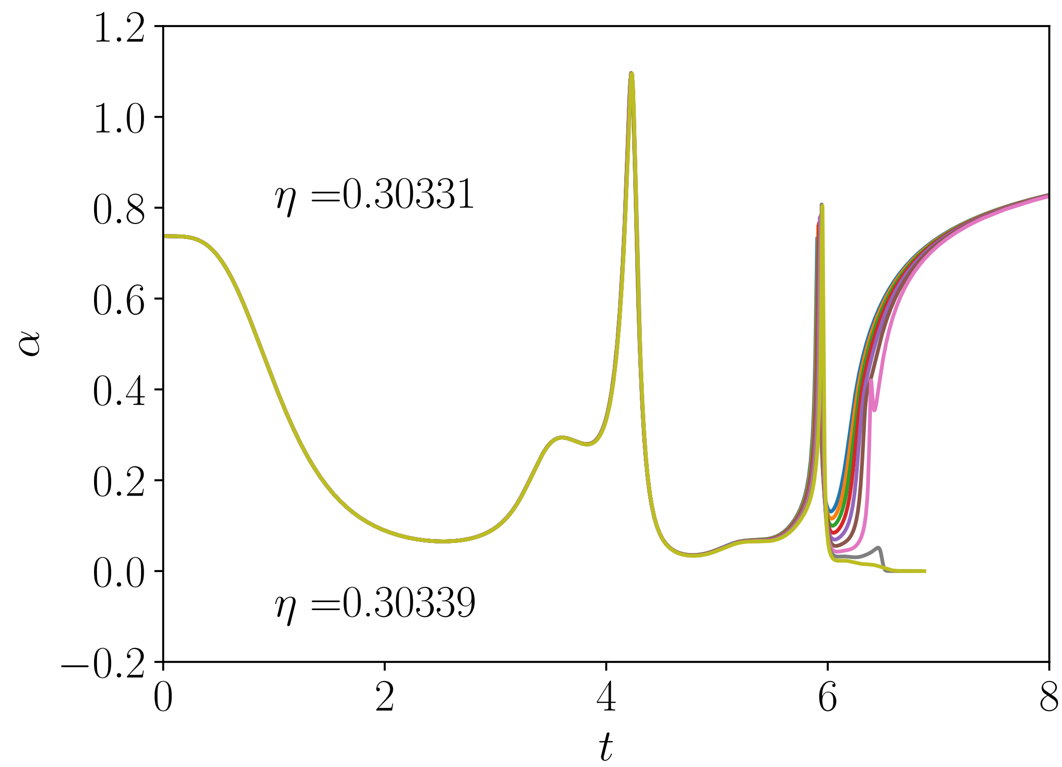
$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



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$$\eta > \eta_* \qquad \alpha \to 0 \qquad \text{end up with black hole}$$

Black-hole threshold

$$0.30337 < \eta_* < 0.30338$$

A numerical experiment...

- Let's say scalar wave

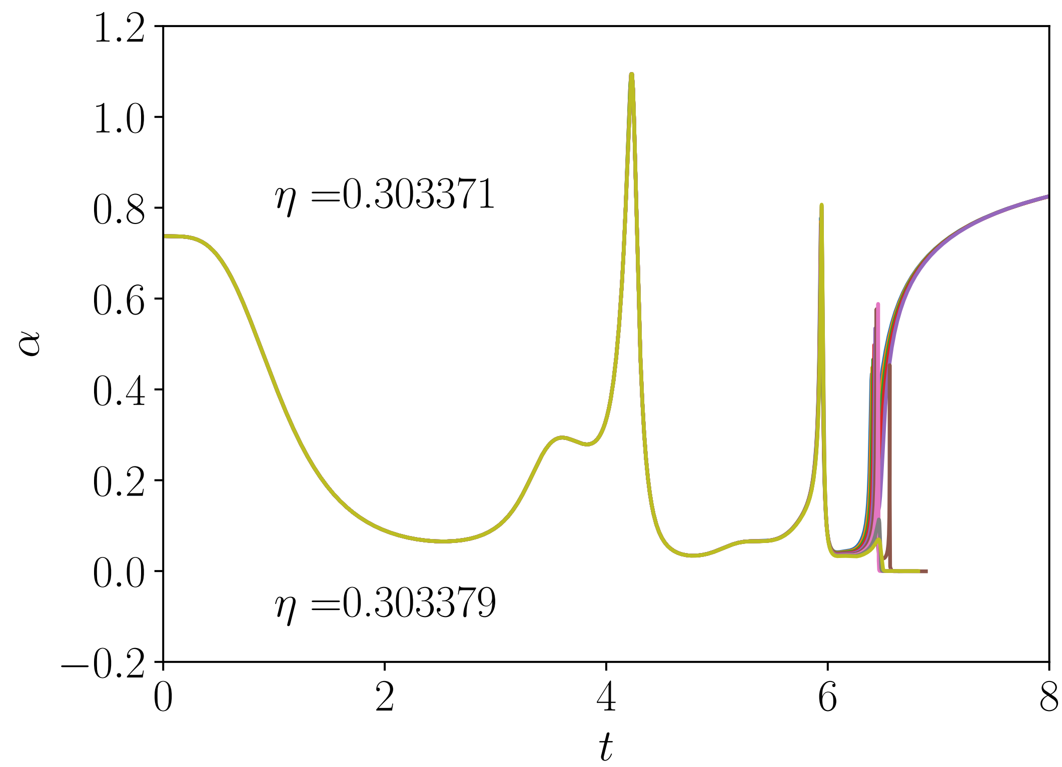
$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



Have *critical value* η_* so that

$\eta < \eta_*$ $\alpha \to 1$ end up with flat space

$\eta > \eta_*$ $\alpha \to 0$ end up with black hole

Black-hole threshold

$$0.303375 < \eta_* < 0.303376$$

A numerical experiment...

- Let's say scalar wave

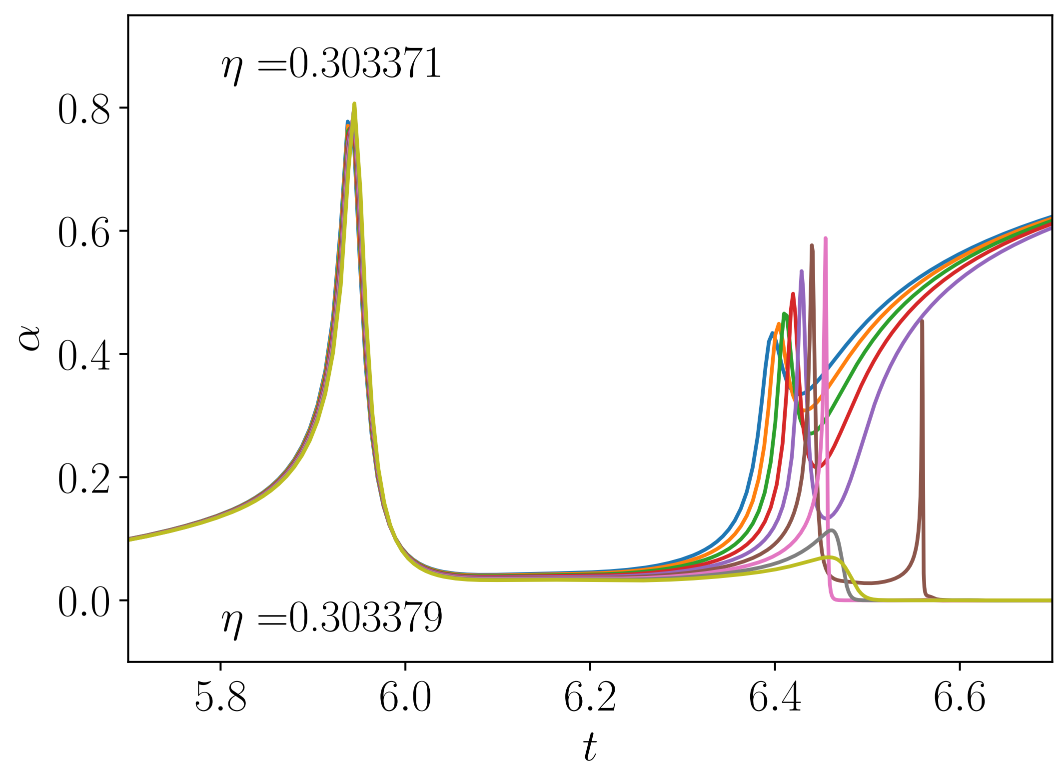
$$\Box\phi \equiv g^{ab}\nabla_a\nabla_b\,\phi = 0$$

coupled to Einstein's equations

- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



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$\eta > \eta_*$ $\quad \alpha \to 0$ $\quad$ end up with black hole

Black-hole threshold

$$0.303375 < \eta_* < 0.303376$$

A numerical experiment...

- Let's say scalar wave

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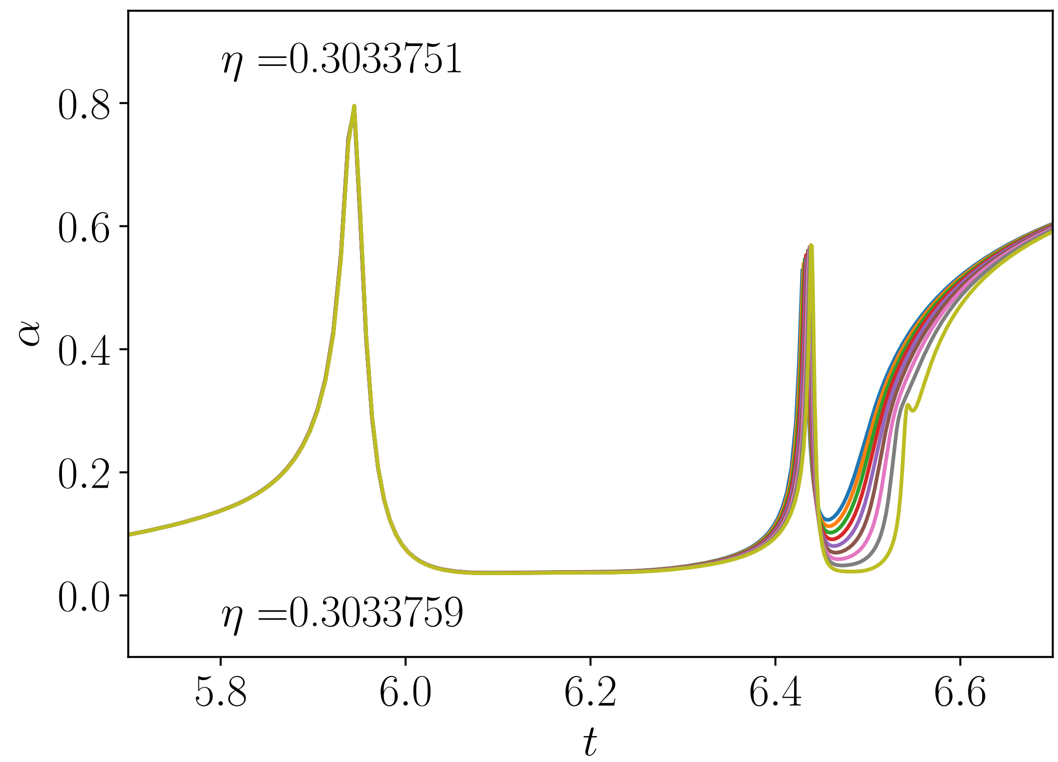

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Black-hole threshold

$$0.3033759 < \eta_* < 0.3033760$$

A numerical experiment...

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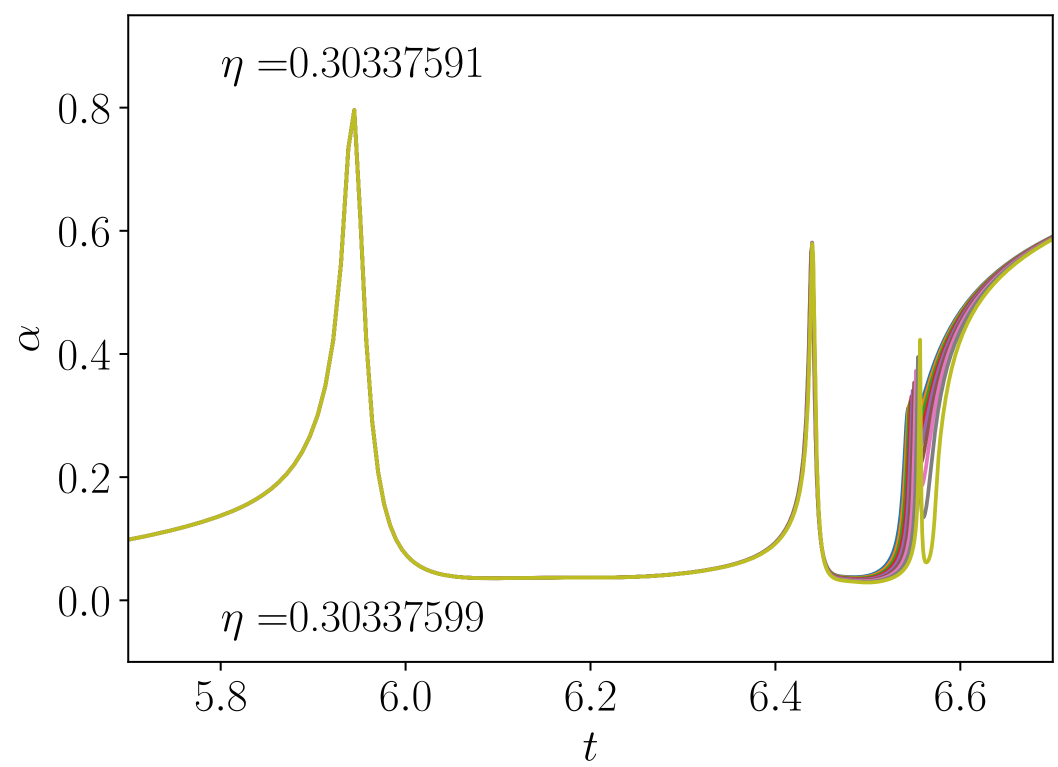

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Black-hole threshold

$$0.30337599 < \eta_* < 0.30337600$$

A numerical experiment...

- Let's say scalar wave

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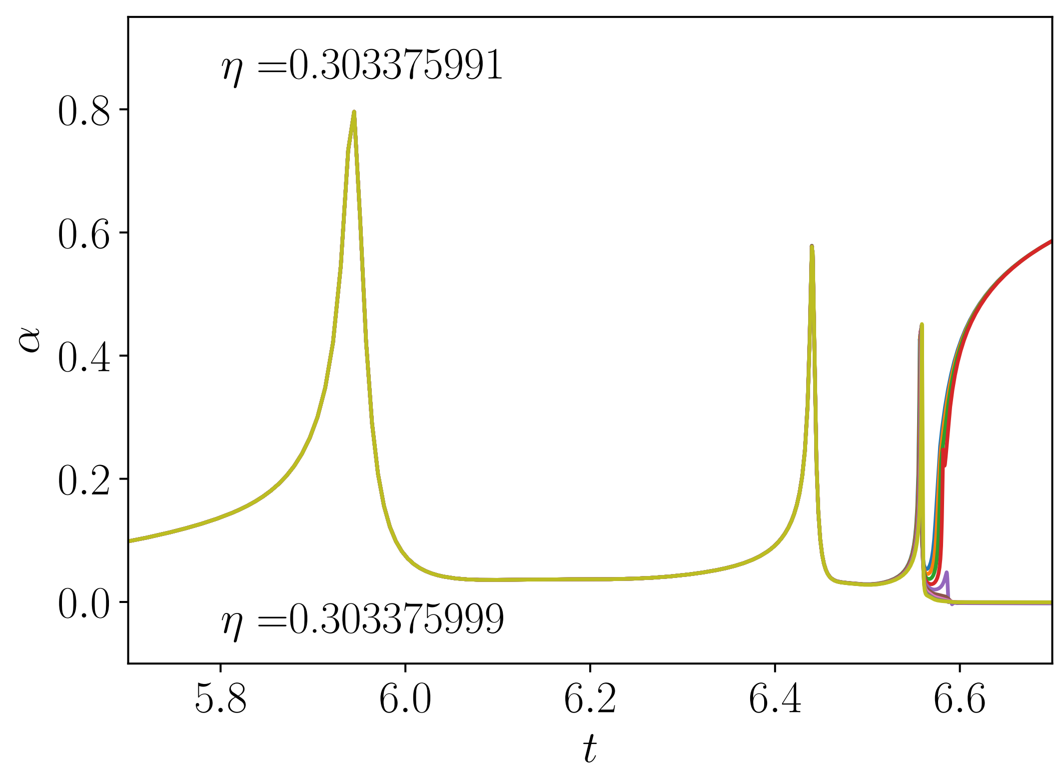
coupled to Einstein's equations

- Initial data

$$\phi = \eta \exp(-R^2/R_0^2)$$

- try out different η...



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Black-hole threshold

$$0.303375994 < \eta_* < 0.303375995$$

A numerical experiment...

- Let's say scalar wave

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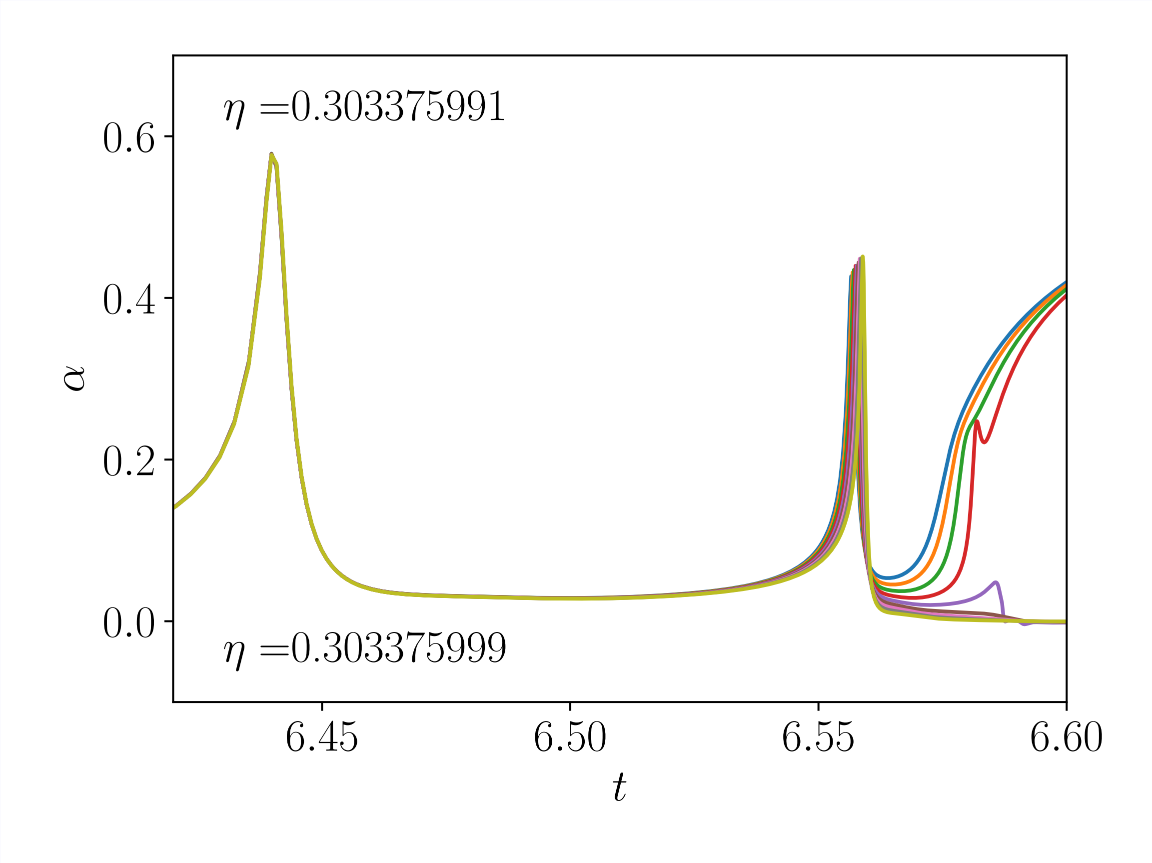

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Black-hole threshold

$$0.303375994 < \eta_* < 0.303375995$$

A numerical experiment...

- Let's say scalar wave

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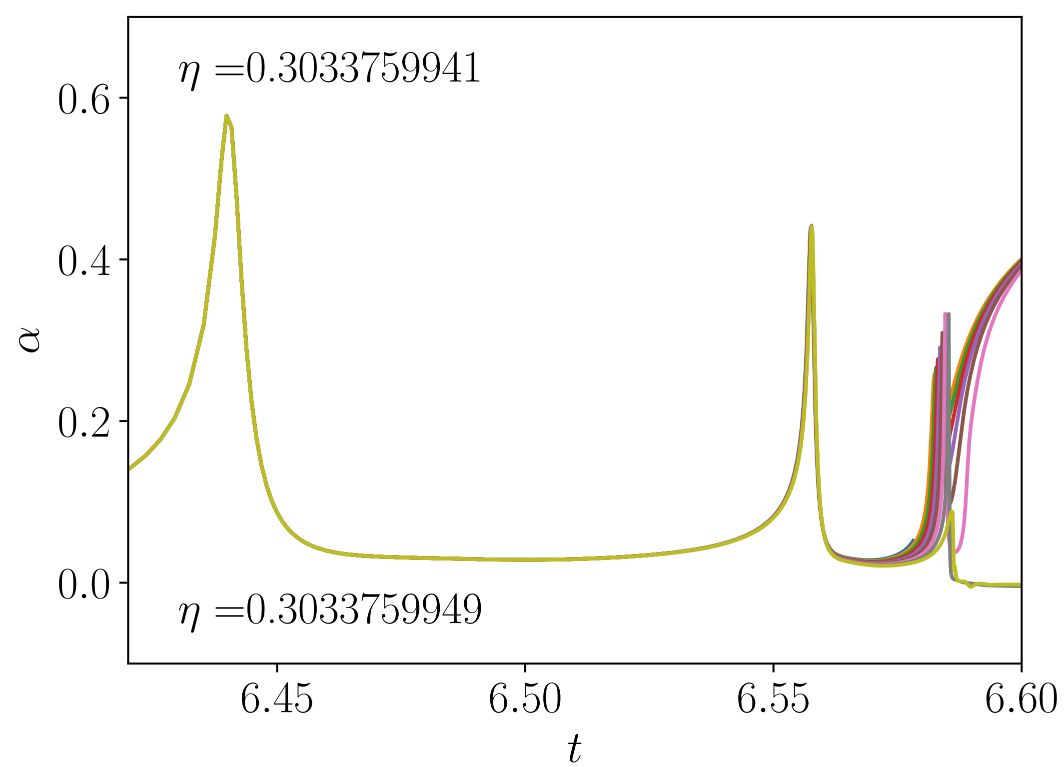

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Black-hole threshold

$$0.3033759947 < \eta_* < 0.3033759948$$

A numerical experiment...

- Let's say scalar wave

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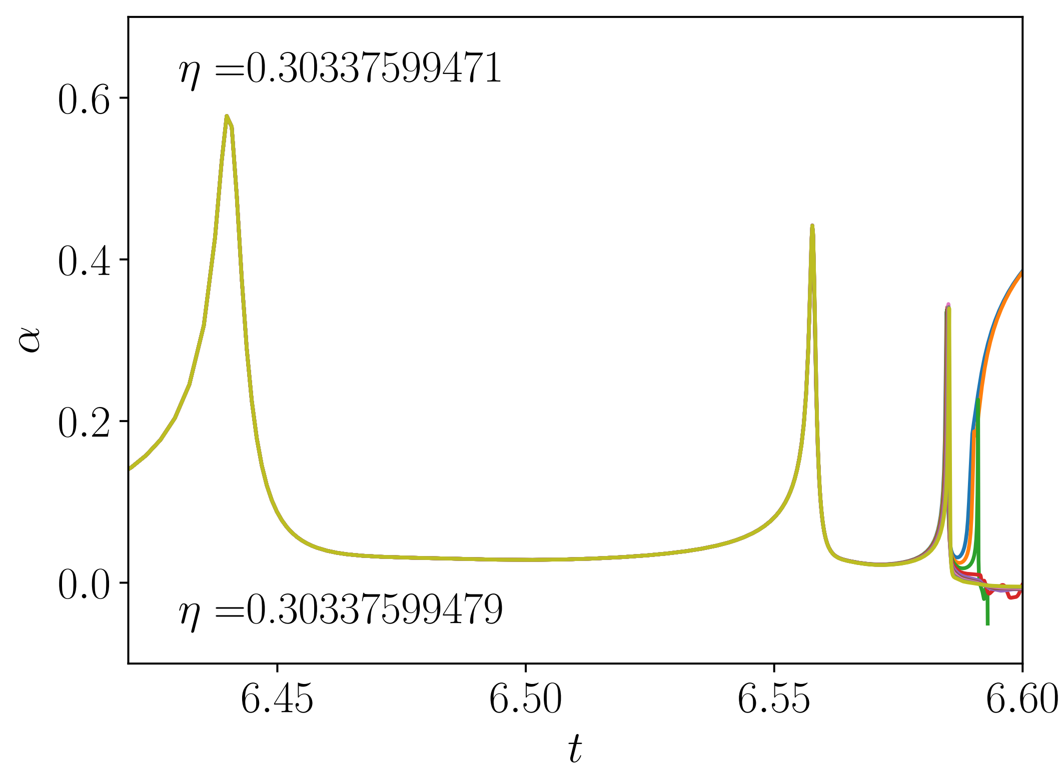
- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



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Black-hole threshold

$$0.30337599472 < \eta_* < 0.30337599473$$

A numerical experiment...

- Let's say scalar wave

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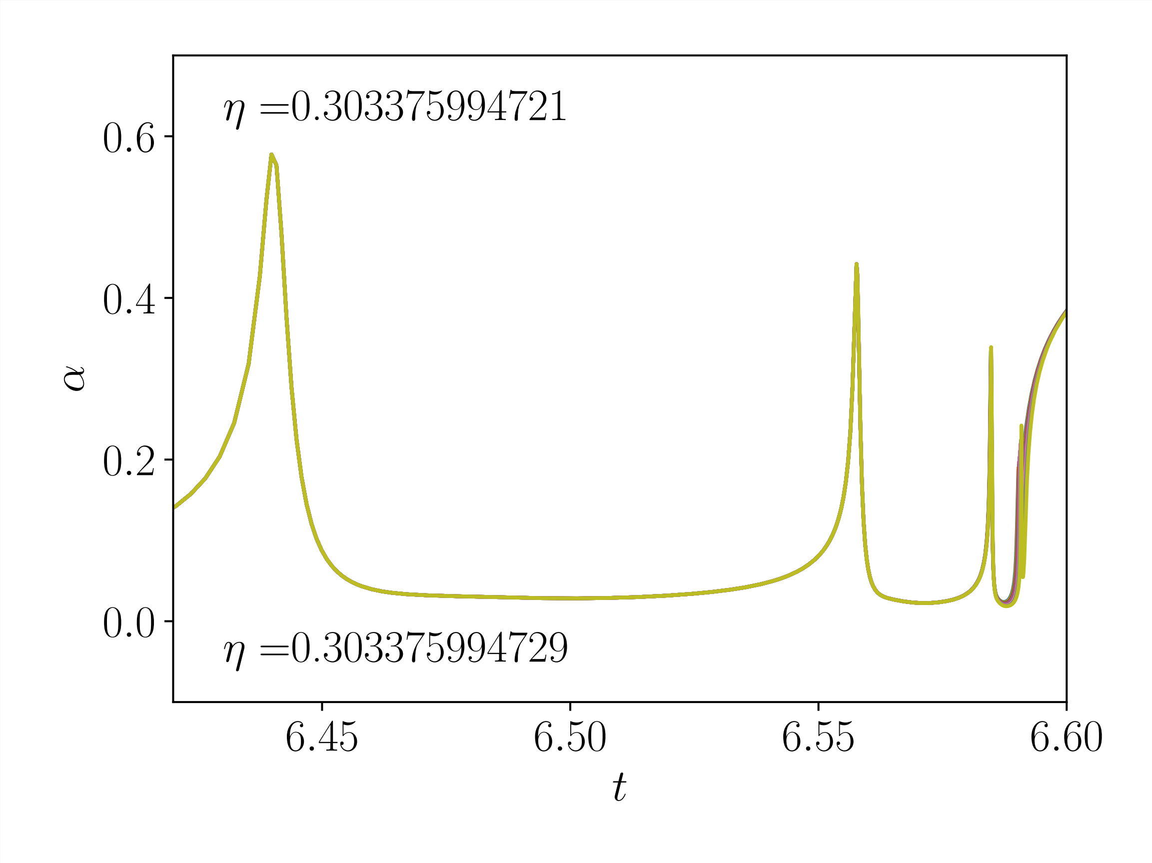
- Initial data

$$\phi = \eta\exp(-R^2/R_0^2)$$

- try out different η...



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Black-hole threshold

$$0.303375994729 < \eta_* < 0.303375994730$$

A numerical experiment...

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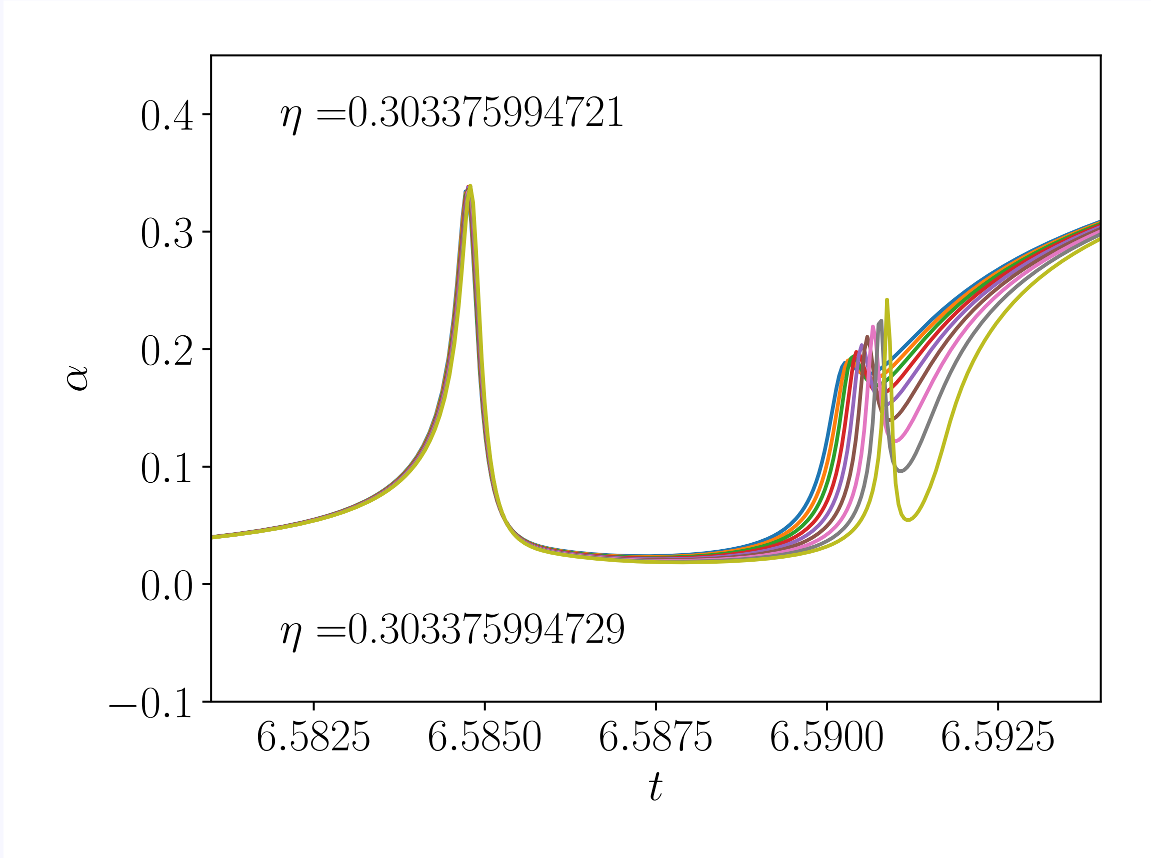

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Black-hole threshold

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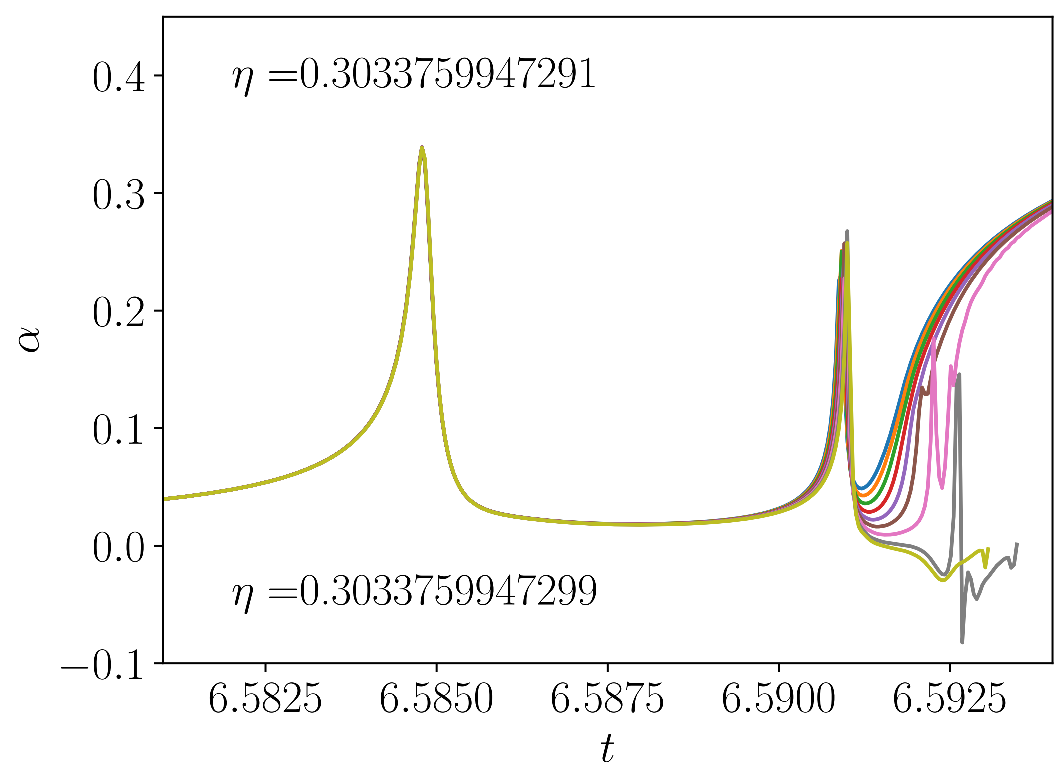

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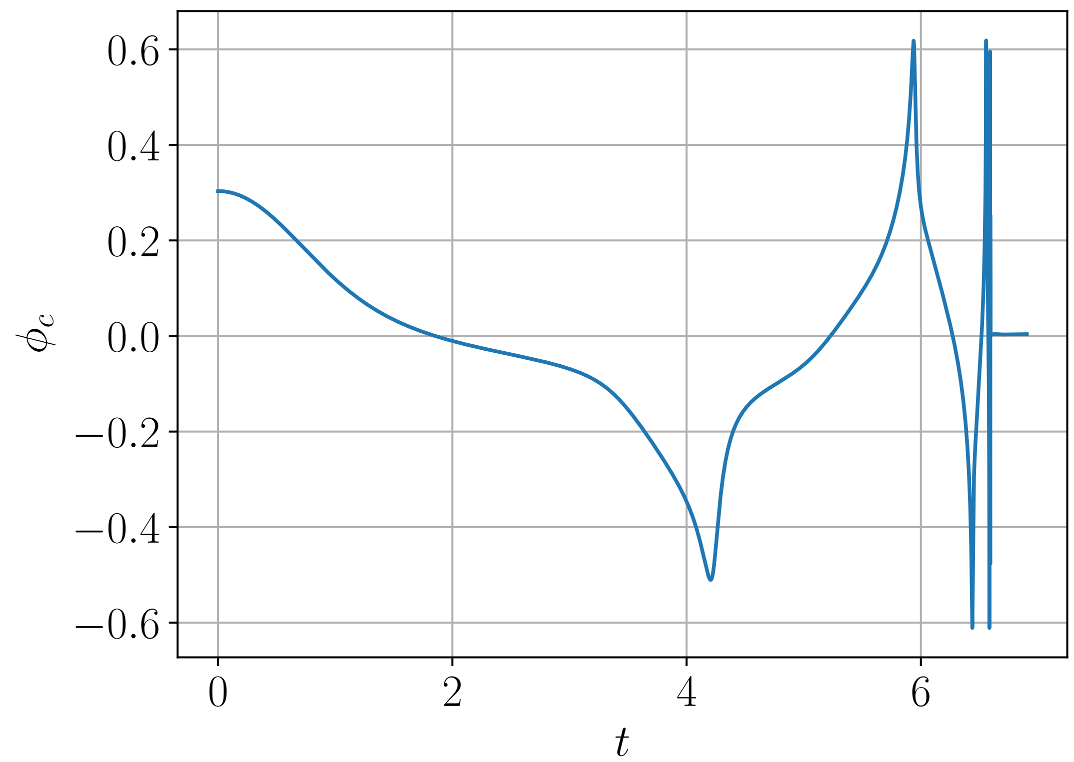
Black-hole threshold

$$0.3033759947297 < \eta_* < 0.3033759947298$$

Critical Solution

- Let's look at ϕ for $\eta \approx \eta_*$ at $r = 0$

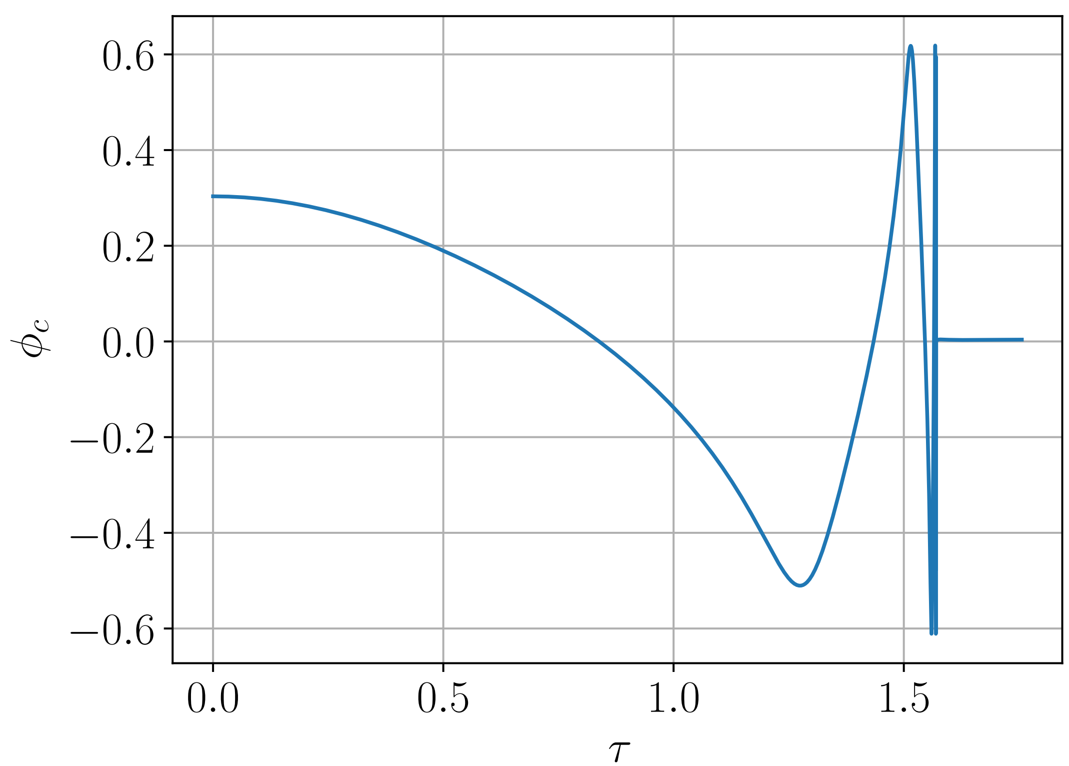


Critical Solution

- Let's look at ϕ for $\eta \approx \eta_*$ at $r = 0$
- plot as function of proper time τ

$\Longrightarrow$ oscillations "accumulate" at "accumulation" time

$$\tau_* \approx 1.5698$$



Critical Solution

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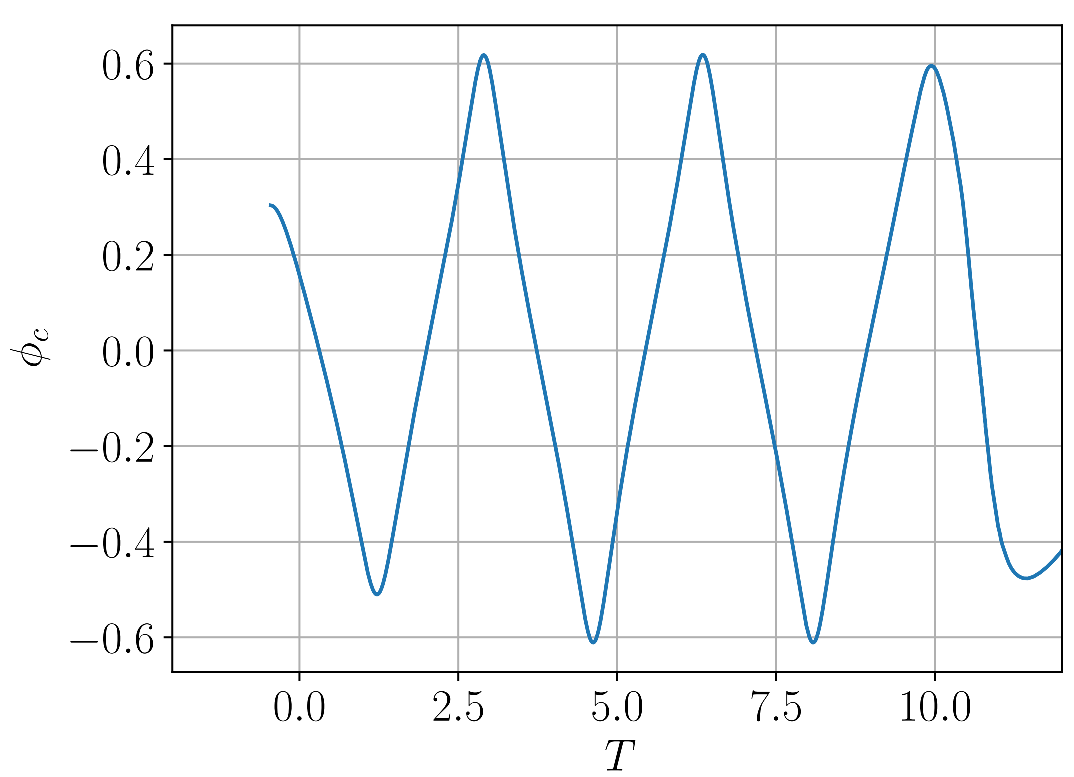
$$\tau_* \approx 1.5698$$

- plot as function of

$$T \equiv -\log(\tau_* - \tau)$$



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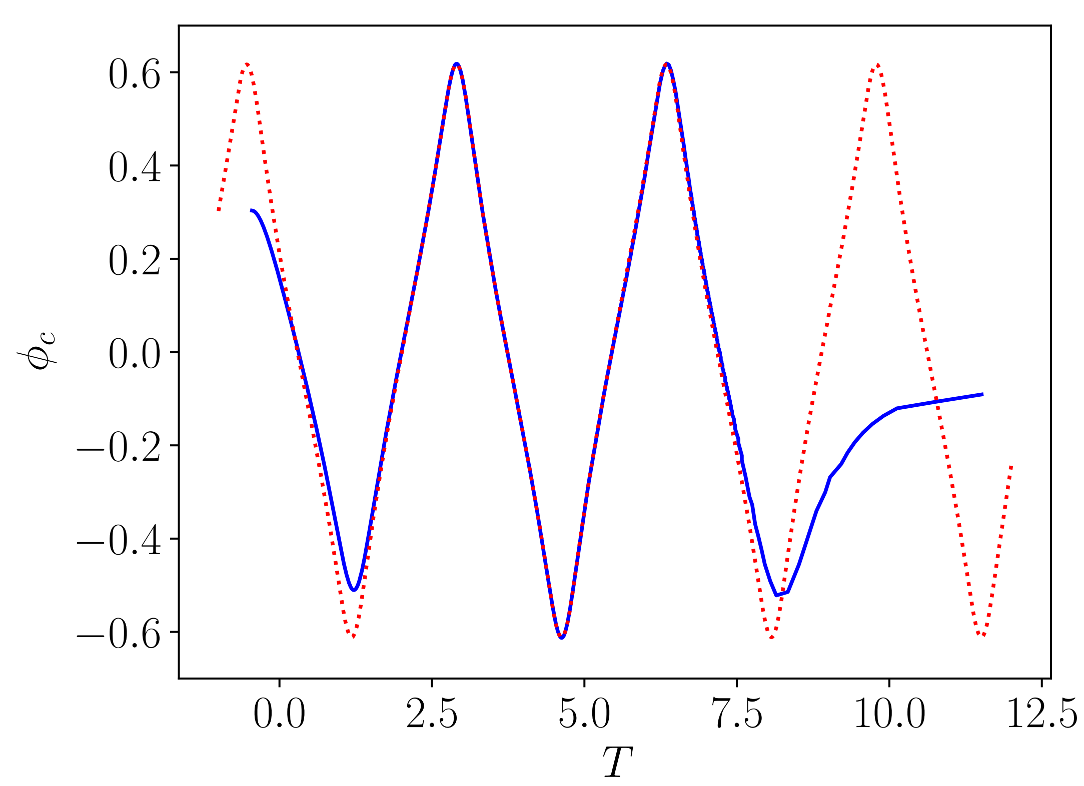


$\Longrightarrow$ critical solution performs periodic oscillations in T (discrete self-similarity)

$\Longrightarrow$ "Choptuik spacetime"

Can we form arbitrarily small black holes?

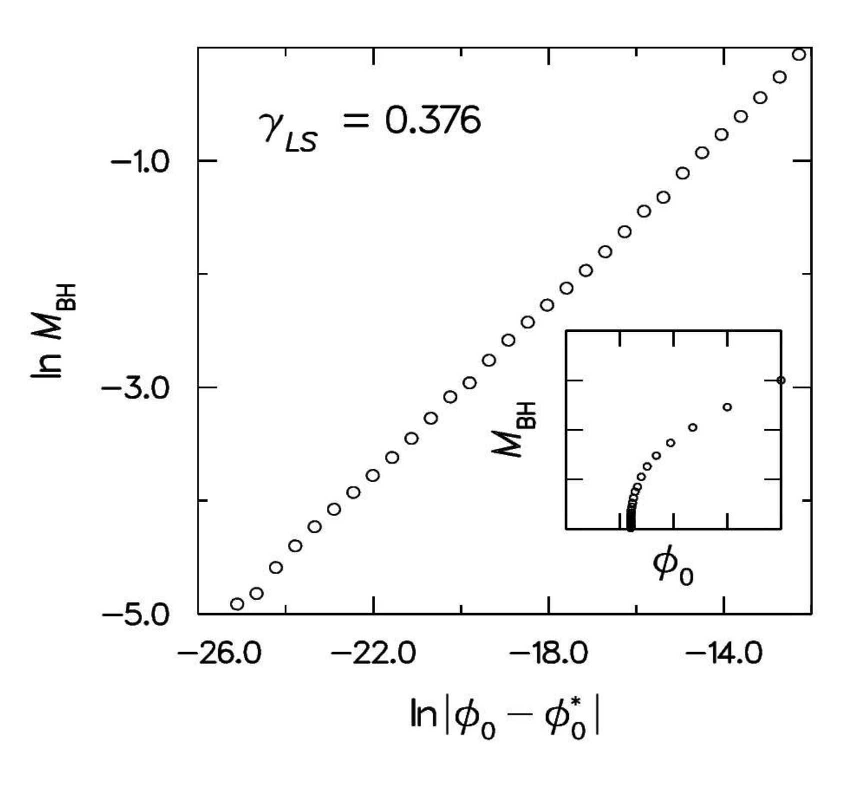
Plot mass M of forming black hole as function of parameter η

$\Longrightarrow$ find power-law scaling

$$M \simeq (\eta - \eta_*)^\gamma$$

with *critical exponent* γ universal (for given matter field)

- reminiscent of critical phenomena in other fields of physics
- can form arbitrarily small black holes



[Choptuik, 1998]

Critical Phenomena in Gravitational Collapse

Consider initial matter distribution parametrized by η (say density) and evolve...

Then critical parameter η_* separates
◦ supercritical data: form black hole
◦ subcritical data: don't

Close to η_* observe *critical phenomena*:

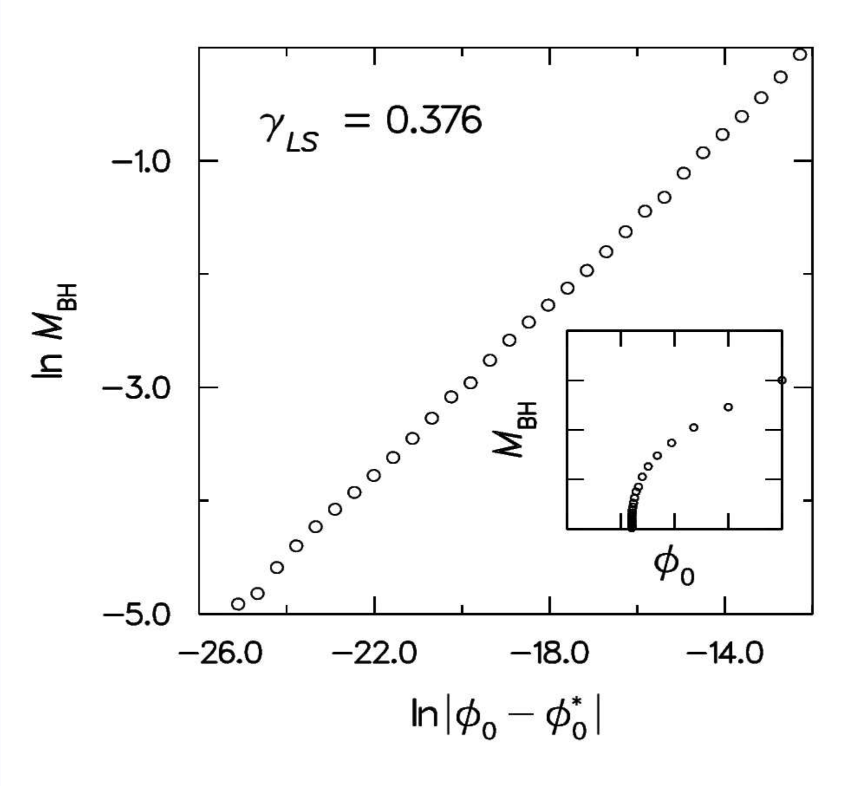
- black hole formed from supercritical data has mass

$$M \simeq |\eta - \eta_*|^\gamma$$

where γ is universal

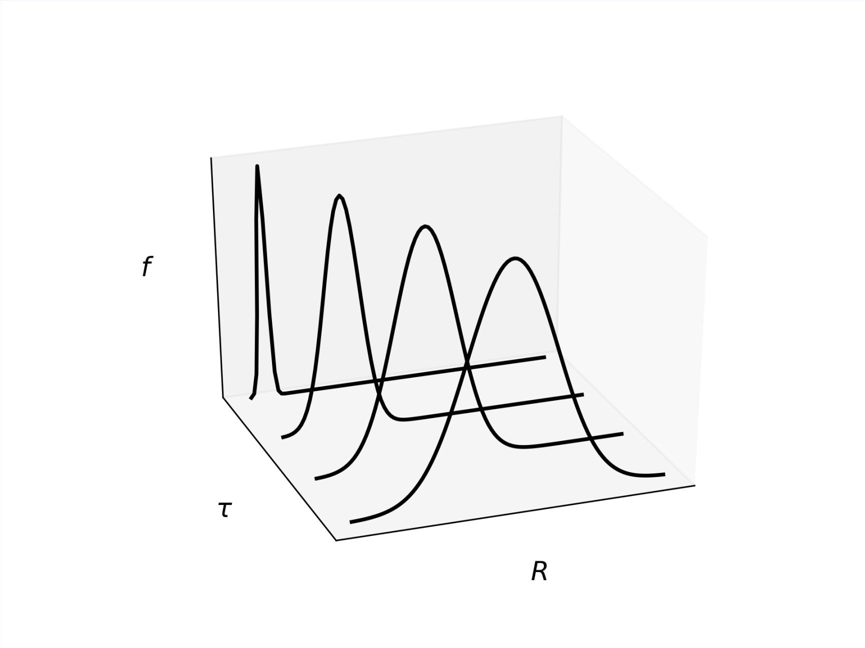
- spacetime approaches self-similar critical solution

[Choptuik, 1993]



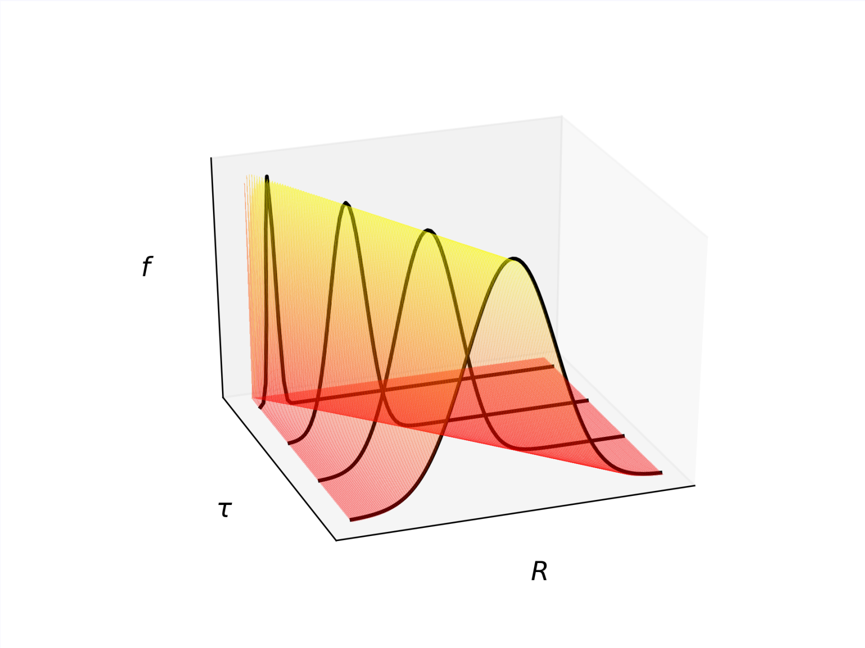
[Choptuik, 1998]

Self-similarity



- Solution contracts without changing shape. . .

Self-similarity



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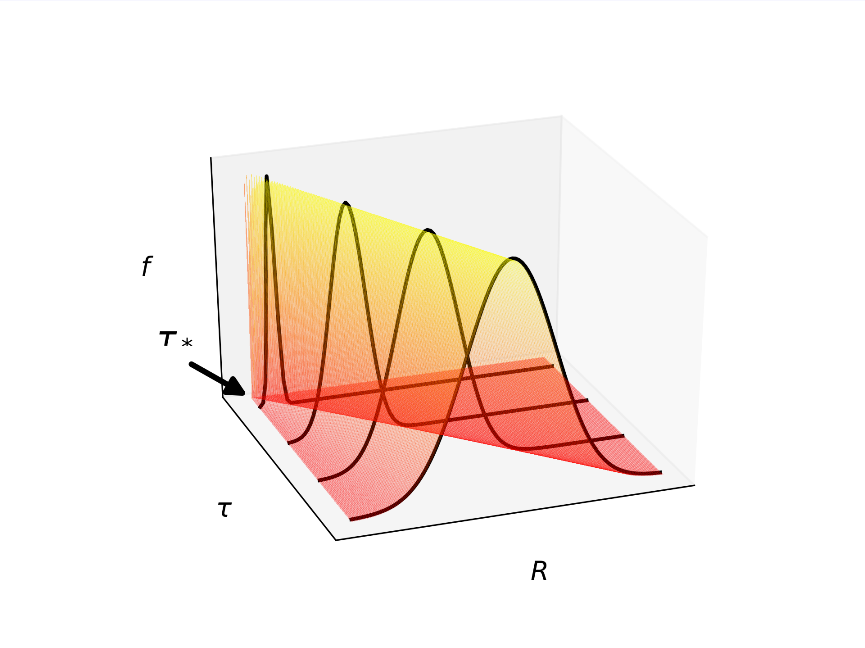
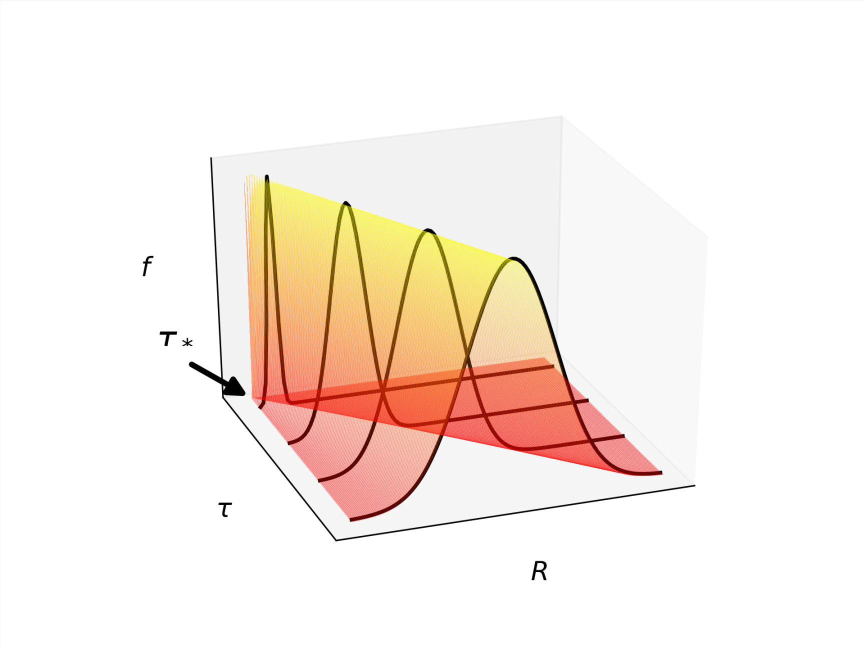


- Solution contracts without changing shape. . .
- . . . towards accumulation event at $\tau = \tau_*$

Self-similarity



- Solution contracts without changing shape. . .
- . . . towards accumulation event at $\tau = \tau_*$
- radius R proportional to $\tau_* - \tau$,

$$R \simeq (\tau_* - \tau)$$

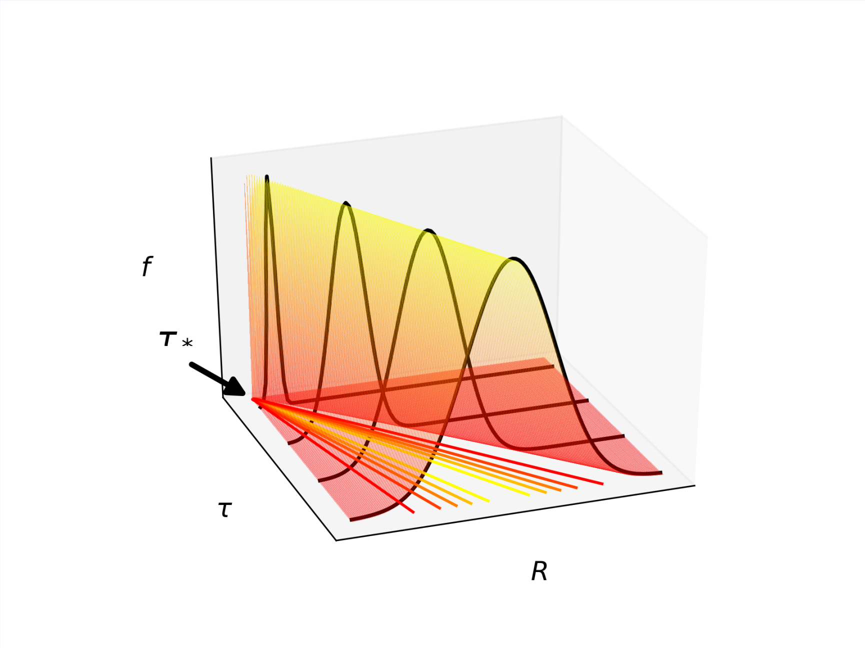
$\Longrightarrow$ dimensionless quantities are functions of

$$\xi \equiv \frac{R}{\tau_* - \tau}$$

only, i.e.

$$Z = Z_*(\xi)$$

Self-similarity



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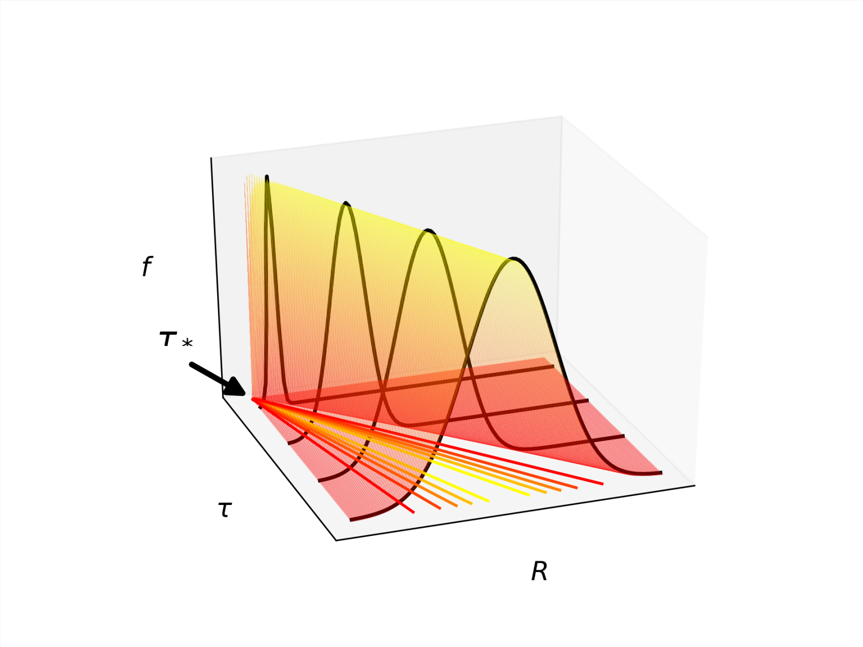
Self-similarity



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$$R \simeq (\tau_* - \tau)$$

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$$\xi \equiv \frac{R}{\tau_* - \tau}$$

only, i.e.

$$Z = Z_*(\xi)$$

$\Longrightarrow$ no preferred global length scale

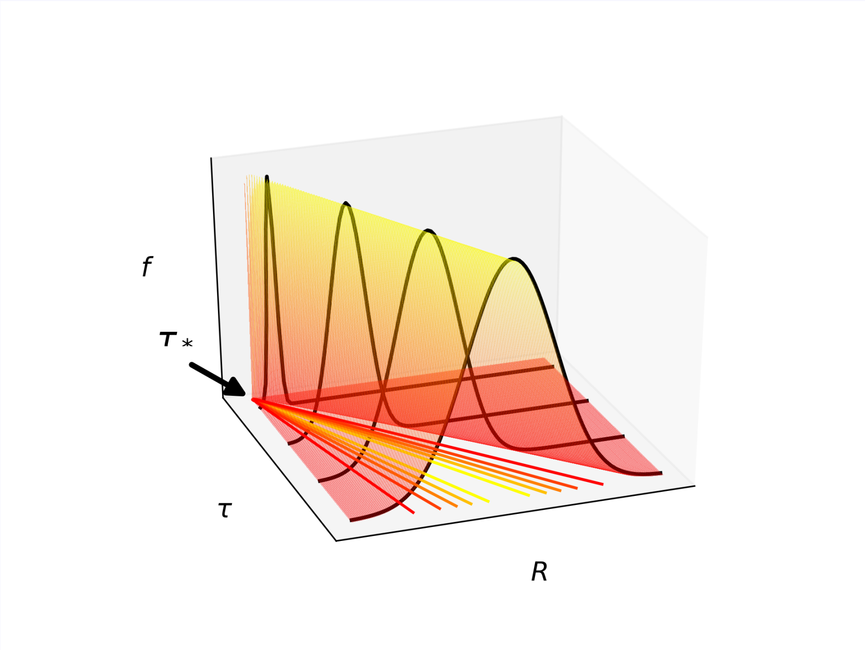
What sets scale of forming black holes?

Three phases of evolution

- Phase I:
  from initial data to something close to critical solution
  (how close? depends on degree of fine-tuning)
- Phase II:
  critical solution plus perturbation
  (until perturbation becomes nonlinear)
- Phase III:
  collapse to black hole or disperse

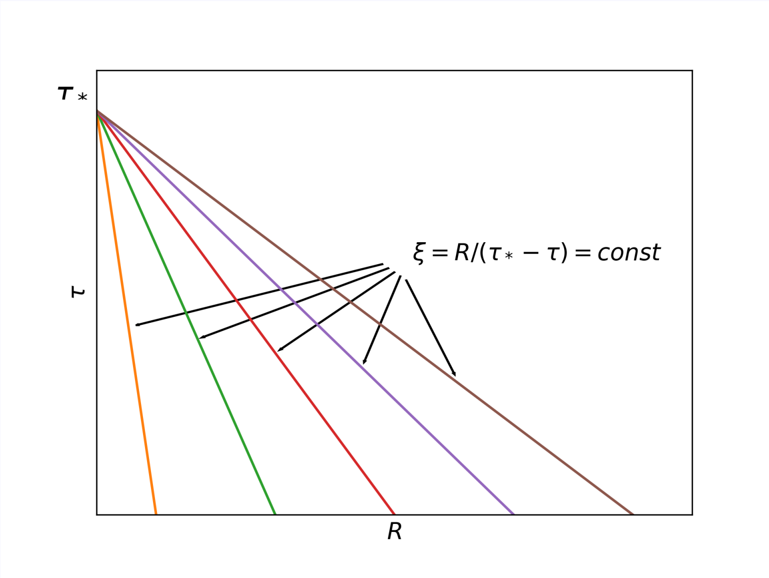
$\Longrightarrow$ length scale set by size of self-similar solution at transition from Phase II to III

Phase II: Perturbations of Critical Solutions



- Consider perturbations ζ of critical solution

Phase II: Perturbations of Critical Solutions



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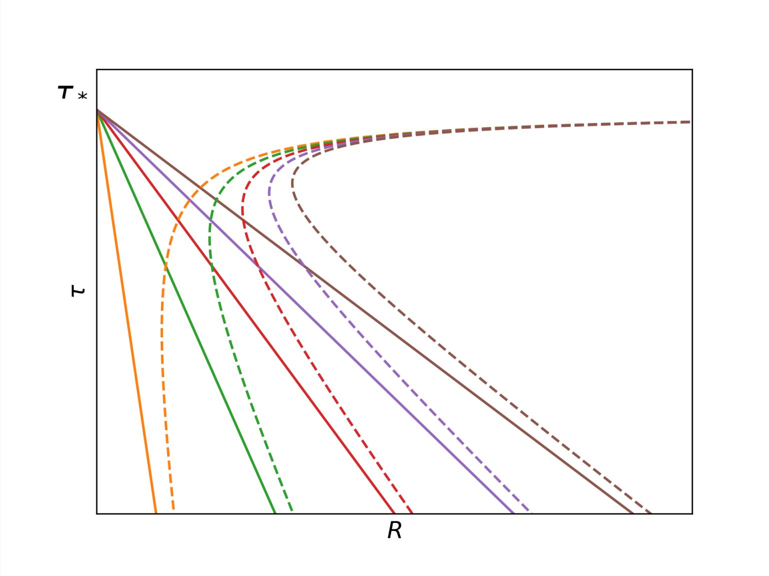
Phase II: Perturbations of Critical Solutions



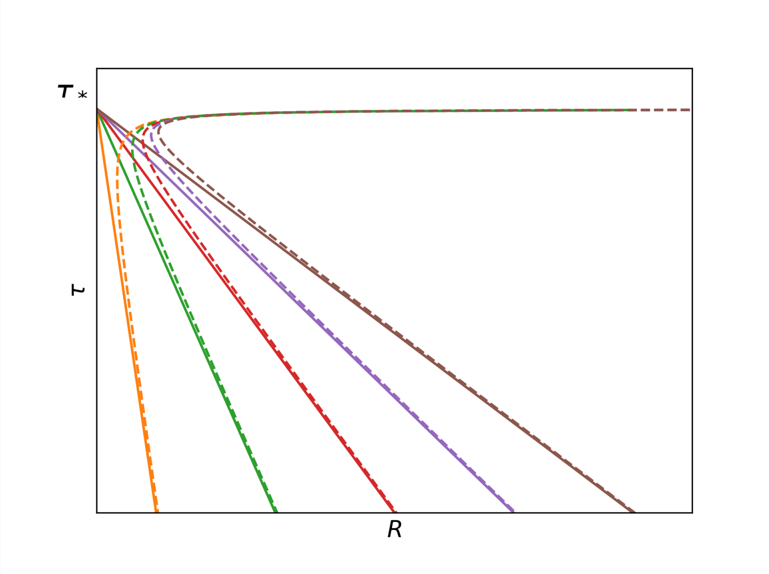
- Consider perturbations ζ of critical solution

- assume that only one mode is unstable

$\Longrightarrow$ grows at rate λ in $T = -\log(\tau_* - \tau)$

$$\zeta \propto \exp(\lambda T) = (\tau_* - \tau)^{-\lambda}$$

Phase II: Perturbations of Critical Solutions



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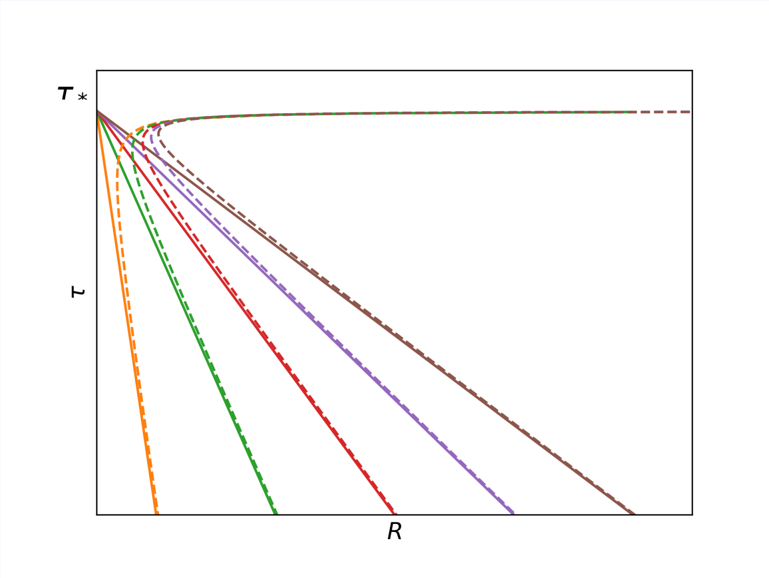
$$\zeta \propto \exp(\lambda T) = (\tau_* - \tau)^{-\lambda}$$

- to leading order also proportional to $\eta - \eta_*$

$$\zeta \propto (\eta - \eta_*)(\tau_* - \tau)^{-\lambda}$$

Phase II: Perturbations of Critical Solutions



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- to leading order also proportional to $\eta - \eta_*$

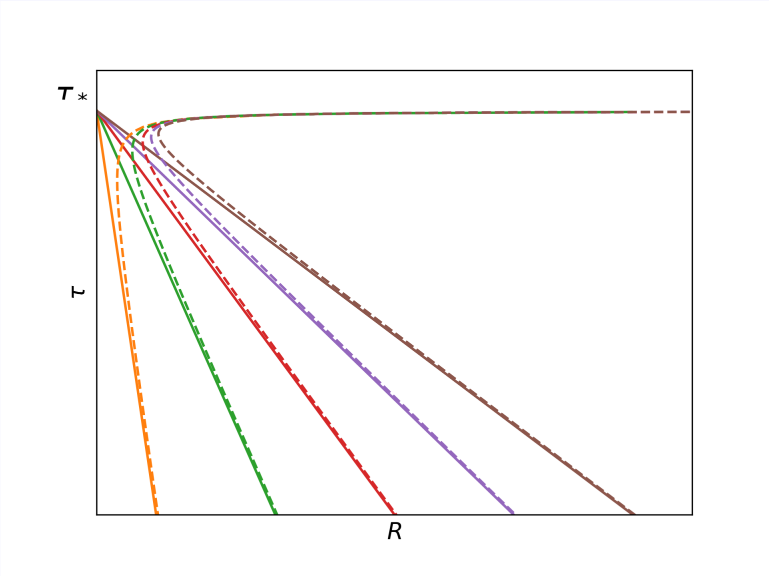
$$\zeta \propto (\eta - \eta_*)(\tau_* - \tau)^{-\lambda}$$

Mode becomes nonlinear when $\zeta = const$

$\Longrightarrow$ determines length scale

$$R \propto (\tau_* - \tau) \propto (\eta - \eta_*)^{1/\lambda}$$

Phase II: Perturbations of Critical Solutions



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- to leading order also proportional to $\eta - \eta_*$

$$\zeta \propto (\eta - \eta_*)(\tau_* - \tau)^{-\lambda}$$

Mode becomes nonlinear when $\zeta = const$

$\Longrightarrow$ determines length scale

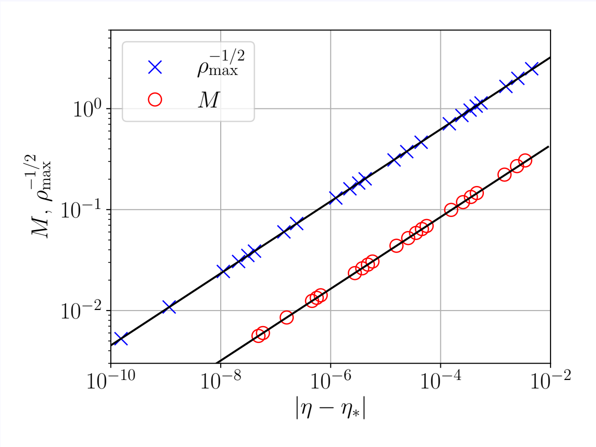
$$R \propto (\tau_* - \tau) \propto (\eta - \eta_*)^{1/\lambda}$$

$\Longrightarrow$ scaling laws, e.g.

$$M \propto (\eta - \eta_*)^{\gamma}$$

with $\gamma = 1/\lambda$

[Koike $et.al.$, 1995; Maison 1995]



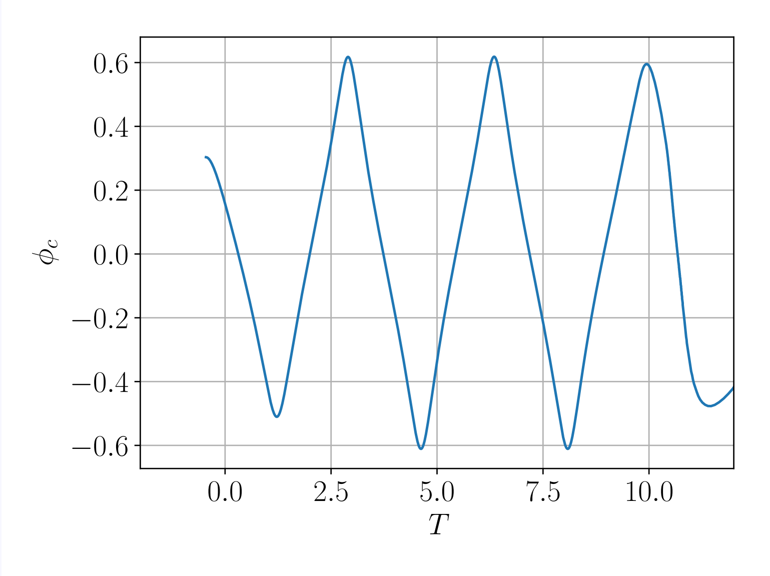
$1/\lambda = 0.3558 \qquad \gamma = 0.356$

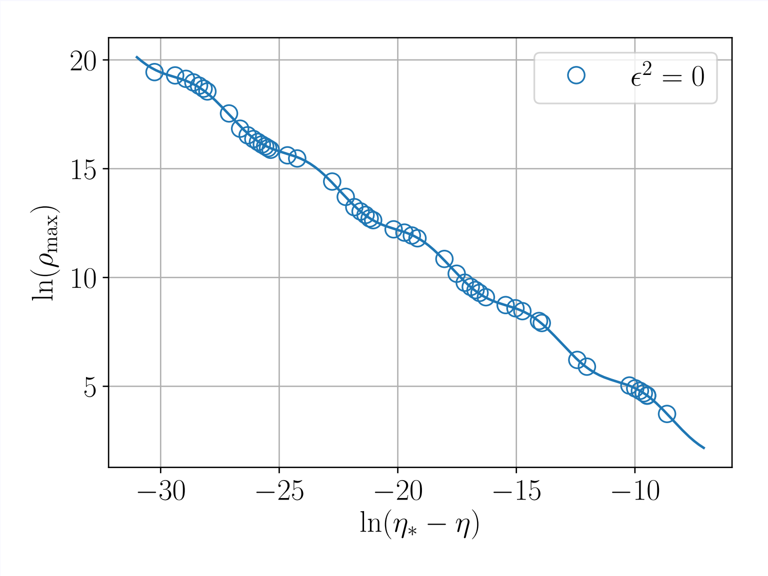
[Celestino & TWB, 2018]

Continuous versus discrete self-similarity

For fluid, for example, encounter *continuous self-similarity* (CSS)

For scalar waves, expect "superimposed" oscillation
$\Longrightarrow$ *discrete self-similarity* (DSS)





$\Longrightarrow$ leaves periodic "wiggle" in power-law scaling
[Gundlach, 1997; Hod & Piran, 1997]

Key ingredients...

- Unique critical solution, either CSS or DSS
- Single unstable mode, Lyapunov exponent λ

$\Longrightarrow$ Power-law scaling with critical exponent $\gamma = 1/\lambda$

- Pretty well established in spherical symmetry...

$\Longrightarrow$ ... but what about non-spherical cases??

Critical collapse of gravitational waves

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Critical Behavior and Scaling in Vacuum Axisymmetric Gravitational Collapse

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(Received 22 December 1992)

We report a second example of critical behavior in gravitational collapse. Collapse of axisymmetric gravitational wave packets is computed numerically for a one-parameter family of initial data. A black hole first appears along the sequence at a critical parameter value $p^\star$. As with spherical scalar-field collapse, a power law is found to relate black-hole mass (the order parameter) and critical separation: $M_{\rm BH} \propto |p - p^\star|^\beta$. The critical exponent is $\beta \simeq 0.37$, remarkably close to that observed by Choptuik. Near-critical evolutions produce echoes from the strong-field region which appear to exhibit scaling.

Numerous attempts to reproduce this...

Despite many attempts...

[e.g. Alcubierre $et.al.$, 2000; Garfinkle & Duncan, 2001; Santamaria, 2006; Rinne, 2008; Sorkin, 2011; Hilditch $et.al.$, 2013; Hilditch $et.al.$, 2017]

... the results of Abrahams & Evans have yet to be reproduced.

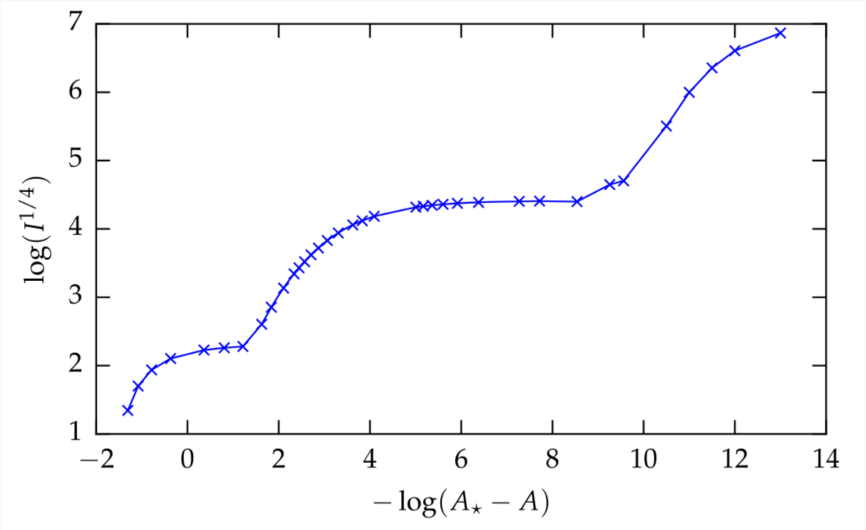
Issues...

- Few of the current 3D numerical relativity codes are designed for critical-collapse simulations
- Some evidence that coordinate conditions that work for other simulations do not work well for critical collapse of gravitational waves

Collapse of Brill waves

- Fine-tune Brill waves to black-hole threshold
- Some agreement with Abrahams & Evans
- But lack of clear evidence for DSS...

[Hilditch, Weyhausen, & Brügmann, 2017]



Critical collapse of electromagnetic waves

Solve Einstein-Maxwell system in axisymmetry

- Forms system of equations similar to that for scalar waves
- Does not allow spherically symmetric critical solution

Consider dipolar initial data of the form

$$E^{\phi} = -\frac{4\eta}{\psi^6}\left(e^{-(r-r_0)^2} + e^{-(r+r_0)^2}\right)$$

Evolve with code that solves BSSN equations in spherical polar coordinates [Baumgarte $et.al.$, 2013]

- "gravitational gauge": 1+log slicing; zero shift
- "EM gauge": choose $\Phi \equiv n^a A_a = 0$

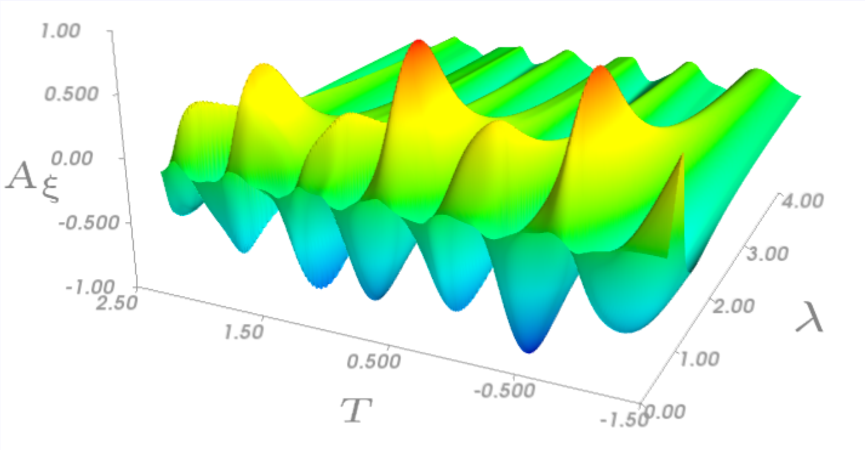
$\Longrightarrow$ fine-tune parameter η to critical value η_*...

[Baumgarte, Gundlach, & Hilditch, 2019]

The critical solution

As invariant diagnostic, consider

$$A_\xi \equiv \frac{\xi^a A_a}{(\xi^a \xi_a)^{1/2}}$$



Here

- A_a electrodynamic vector potential
- $\xi^a = \partial/\partial\varphi$ axisymmetric Killing vector
- $T = -\ln(\tau_* - \tau)$
- λ affine parameter along null geodesics

$\Longrightarrow$ *approximate* DSS, with period $\Delta \simeq 0.55$ – but not exact

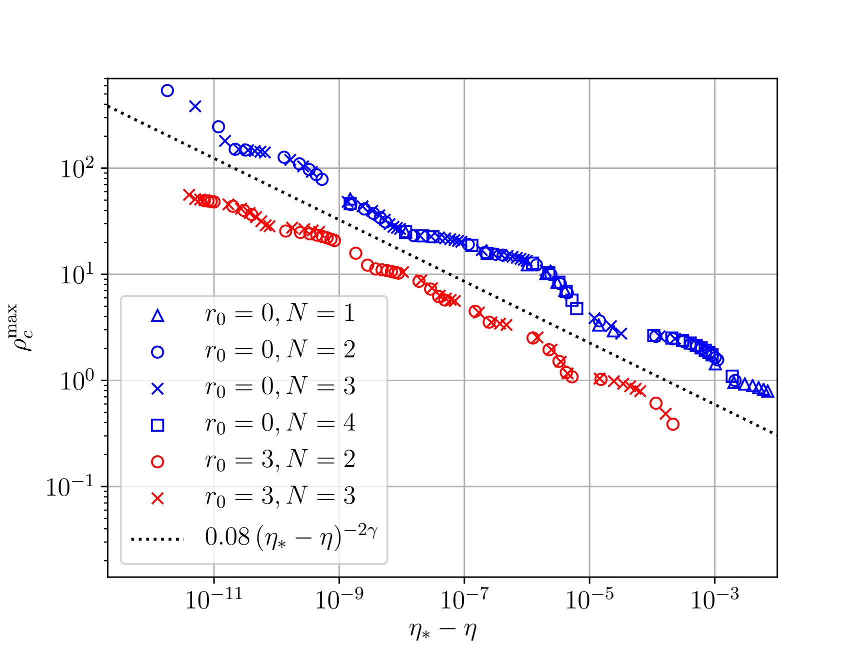
Scaling

- *Approximate* scaling

$$\rho_c^{\max} \simeq (\eta_* - \eta)^{-2\gamma}$$

with $\gamma = 0.145$ – but not exact

- wiggles not exactly periodic



Scaling

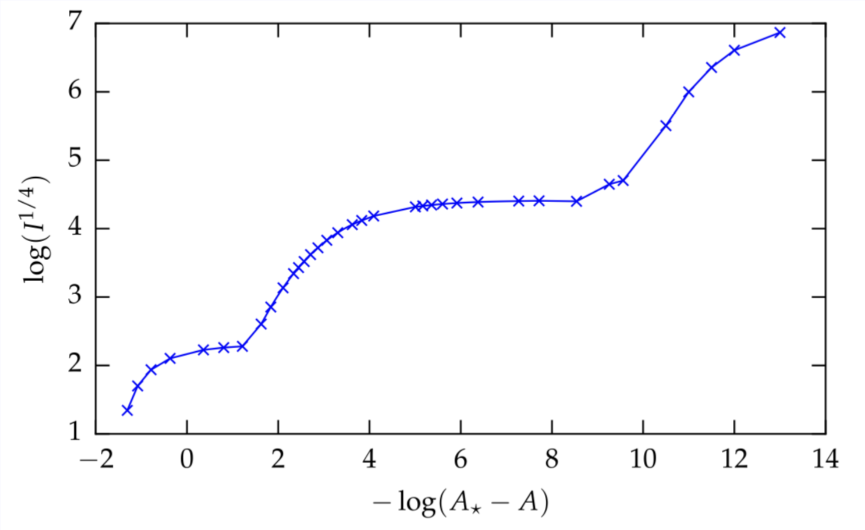
- *Approximate* scaling

$$\rho_c^{\max} \simeq (\eta_* - \eta)^{-2\gamma}$$

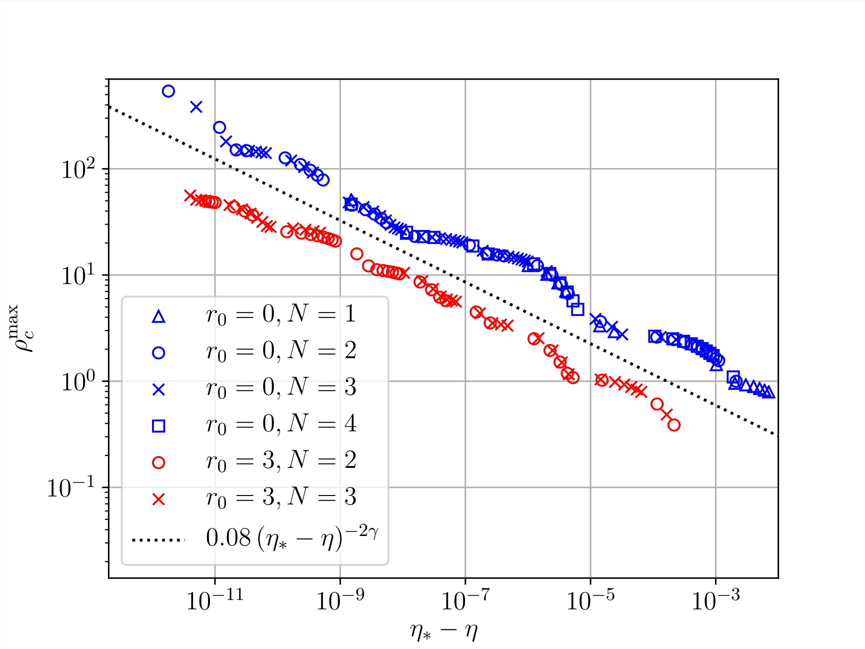
with $\gamma = 0.145$ – but not exact

- wiggles not exactly periodic

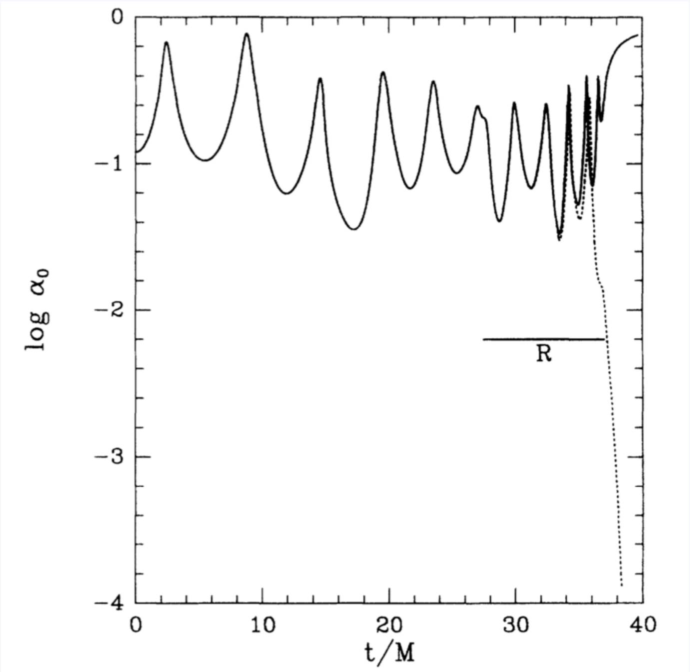
$\Longrightarrow$ reminiscent of



[Hilditch *et.al.*, 2017]

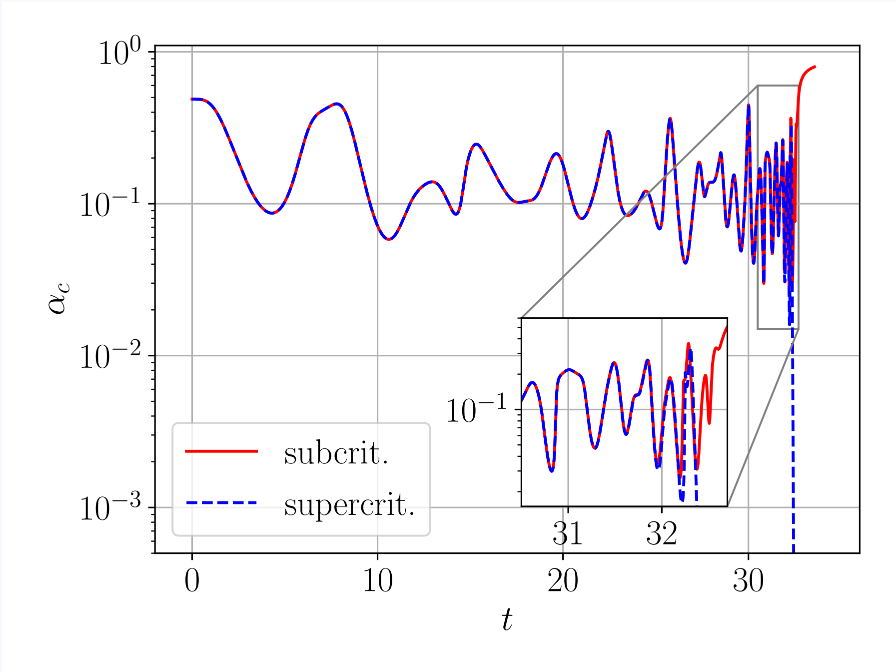


Behavior of lapse



Gravitational waves

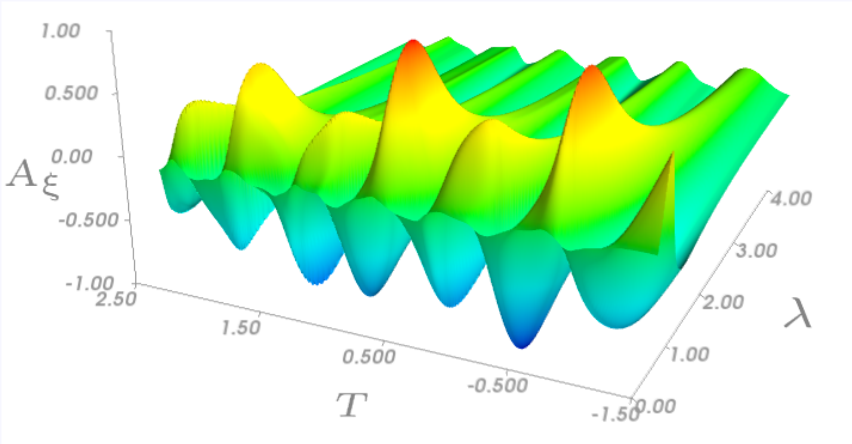
[Abrahams & Evans, 1994]



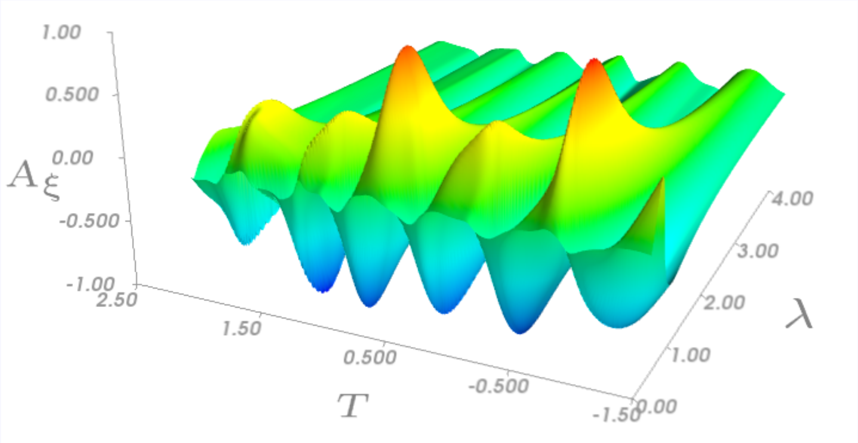
Electromagnetic waves

$\Longrightarrow$ No conclusive evidence for strict periodicity in either case

Is the critical solution unique?



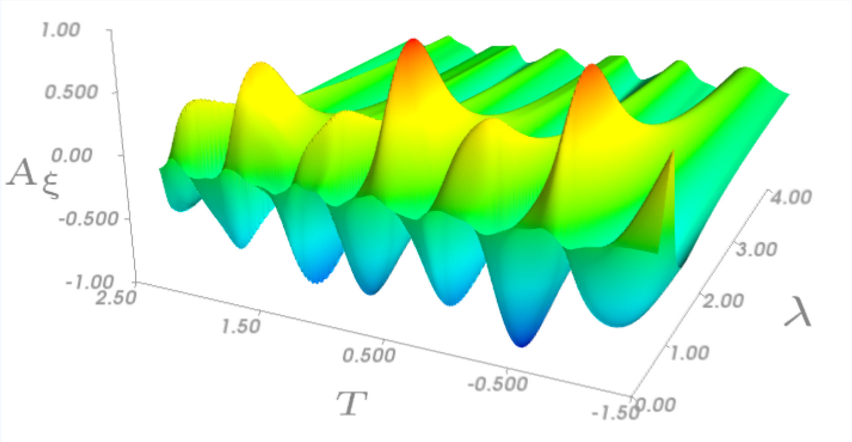
Centered $(r_0 = 0)$



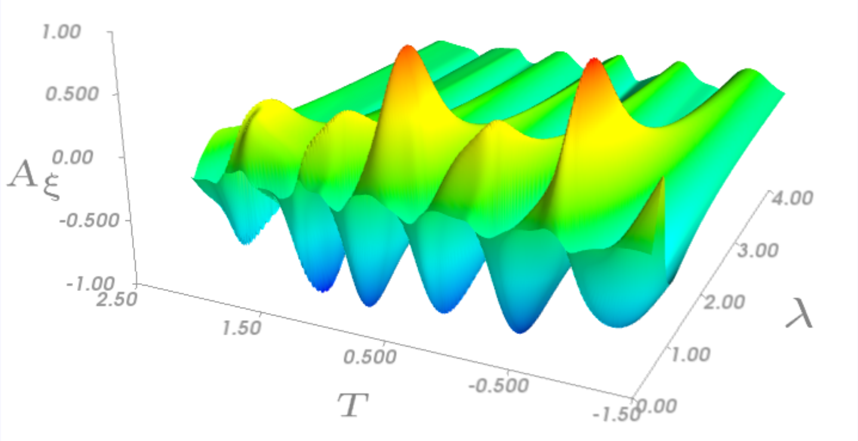
Off-centered $(r_0 = 3)$

$\Longrightarrow$ No evidence for strict uniqueness

Is the critical solution unique?



Centered $(r_0 = 0)$



Off-centered $(r_0 = 3)$

$\Longrightarrow$ No evidence for strict uniqueness

$\Longrightarrow$ Considering more general initial data suggests non-uniqueness of critical solution [Perez Mendoza $et.al.$, in prep]

Summary

- Numerical simulations of critical collapse of electromagnetic waves suggest...
  - ... approximate, but not exact DSS of critical solution
  - ... approximate, but not exact power-law scaling
  - ... similarities with results for gravitational waves

- Absence of exact DSS and scaling might be caused by...
  - ... interplay between gravitational and electromagnetic degrees of freedom [Gundlach $et. al.$, 2019]
  - ... interplay between different multipole moments

  $\Longrightarrow$ appear to be related to non-spherical nature of critical solution

- No evidence for uniqueness of critical solution [Fernández $et. al.$, 2020]

> Our notion of critical phenomena in gravitational collapse invokes the existence of a unique, strictly self-similar critical solution with a single unstable mode. This notion does not appear to apply for electromagnetic (or gravitational) waves.